$f:\mathbb{R}\to\mathbb{R}$ — nice, has left-right limits,

finite # of discontinuity.

Euler – Schanuel integral

$$\oint f\,dx = \sum_{x\in\mathbb{R}} \left[f(x) - \frac{1}{2} f(x^+) - \frac{1}{2} f(x^-)\right]$$

all but finitely many elements of $\mathbb{R}$ are zero.

For $f:\mathbb{R}^n \longrightarrow \mathbb{R}$

$$\oint f(x_1,\dots,x_n)\,dx_1$$

If this is also "nice", can integrate wrt x_2 again, etc.

$$\oint\cdots\oint f(x_1,\dots,x_n)\,dx_1 \cdots dx_n.$$

~~Choose~~ This uses the standard basis.

Can choose another basis.

If for every basis, the iterated integral exist, then the linear combination also has iter. int., i.e. they form a vector space.

Ex: $f = \mathbb{1}_E$, $E \subseteq \mathbb{R}^n$.

If $E = \{a\} \subseteq \mathbb{R}$, $\int_\varepsilon f = 1$.

$E = (a,b)$ $\quad \int_\varepsilon f = -1$.

$E = [a,b]$ $\quad \int_\varepsilon f = 1$.

Convex body $K \subseteq \mathbb{R}^n$, cpt convex set.

$$I_\varepsilon := \int_\varepsilon \cdots \int_\varepsilon \mathbb{1}_K \, dx_1 \cdots dx_n .$$

Fix $(x_2, \dots, x_n) \in \mathbb{R}^{n-1}$.

Consider $\int_\varepsilon \mathbb{1}_K \, dx_1 = \mathbb{1}_{\pi_1(K)}$.

Iterate. Then $I_\varepsilon = 1$.

Now consider open convex body.

relative interior K° / U – open convex body

$$I_\varepsilon := \int_\varepsilon \mathbb{1}_U \, dx_n \cdots dx_1$$

$$\int_\varepsilon \mathbb{1}_U \, dx_n = (-1)^{\dim U - \dim \pi_n(U)} \mathbb{1}_{\pi_n(U)} .$$

$I_\varepsilon = (-1)^{\dim U}$.

Let $v = \{v_1, \dots, v_n\}$ ordered basis

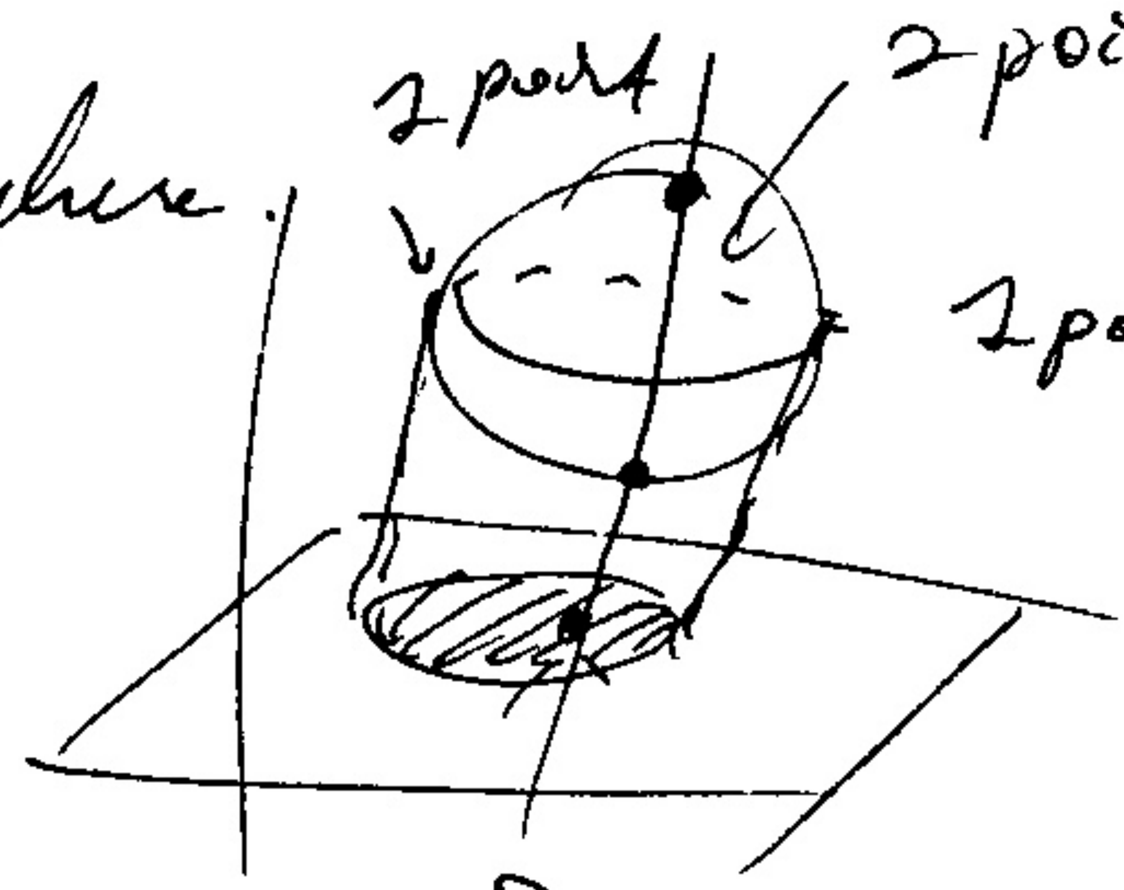
$g(x_1, \dots, x_n) = f(x_1 v_1 + \dots + x_n v_n)$

$\int g\, dx_1 \cdots dx_n = \chi(f, v)$.

Euler ~~characteristic~~ number for convex ~~bodies~~ polytopes.

$V - E + F = 2 \quad \chi(\partial P)$.

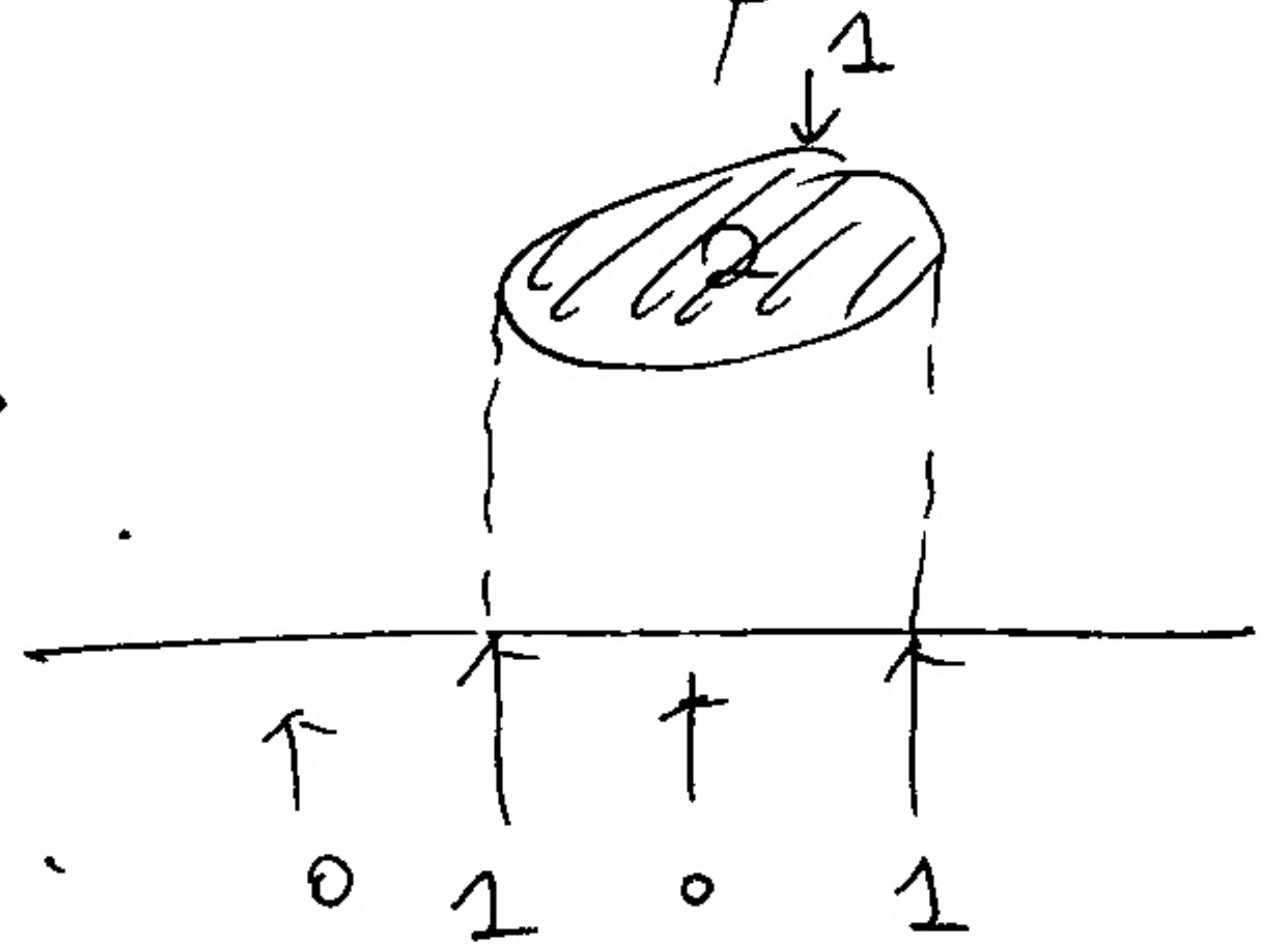
Let S^2 denote the unit sphere.



$\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$.

$g(p) = \int \mathbb{1}_{S^2}\, dx_3$ on π

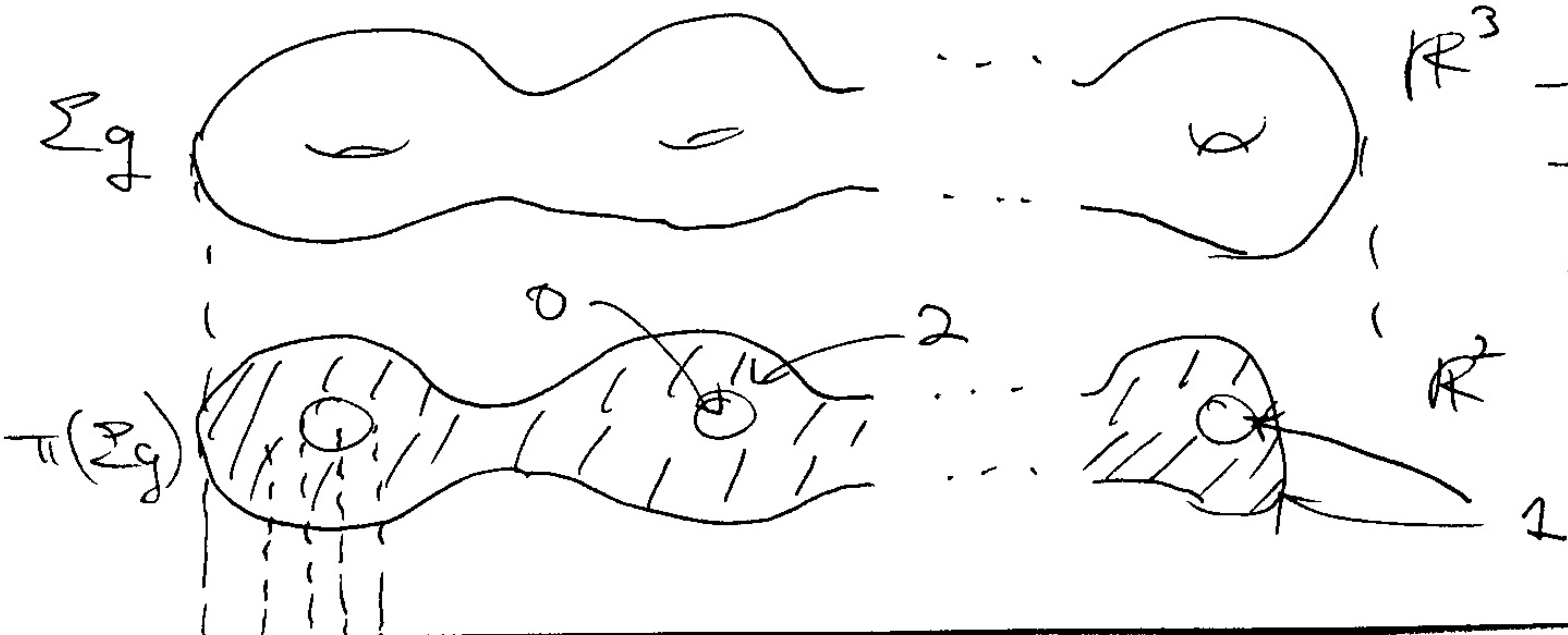
$$= \begin{cases} 1 & p \in \partial(\pi(S^2)) \\ 2 & p \in (\pi(S^2))^\circ \end{cases}.$$



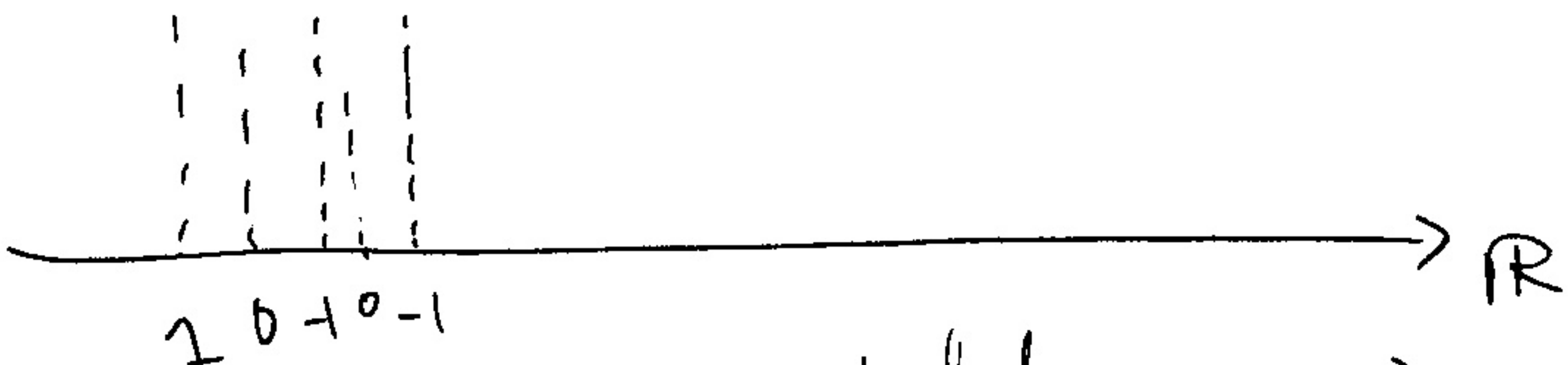
$= \mathbb{1}_{\overline{B^2}} + \mathbb{1}_{B^2}$.

$\int \mathbb{1}_S\, dx_1\, dx_2\, dx_3 = 2 = \chi(S^2)$.

Higher genus bodies.



— Morse Theory related to this



So each hole contributes −2, with the two ends each contributing 1.

So $\oint 1_{\Sigma_g} \, d\vec{x} = 2-2g = \chi(\Sigma_g)$.

End appetiser. Start dinner.

The formal treatment is O-minimal set.
But Bedang thinks we are too stupid.

Let $f: \mathbb{R}^n \to \mathbb{R}$ affine linear.

$E(f>0) = \{x \in \mathbb{R}^d : f(x) > 0\}$ — open half-space

$E(f \geq 0)$ — = — closed half-space.

Take union, intersection, complement finitely many times to produce polyhedrons.

<u>Def</u> Polyhedrons in $\mathbb{R}^d$ are subsets obtained from half-spaces under finite unions, intersections, and complements.

Polyhedrons form a Boolean algebra. $B(\mathcal{P}^d)$

— What if f is replaced by polynomials?
half-spaces become semi-algebraic sets.

Let $X \in$ ~~$\mathcal{B}(\mathbb{P}^d)$~~ $\mathcal{B}(\mathbb{P}^d)$, what is $\fint 1_X$?

To treat ~~such~~ such possibly unbounded sets, we want to instead consider $\overline{\mathbb{R}}$.

$$\fint_{:= \bigcup_{d \in \mathbb{N}} \mathbb{P}^d} 1 f \, dx = \sum_{x \in \overline{\mathbb{R}}} \left[f(x) - \tfrac{1}{2} f(x^+) - \tfrac{1}{2} f(x^-) \right].$$

$\mathbb{P}^{\cancel{d}}$ forms a category, with morphisms

$f : X \longrightarrow Y$, where $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$

s.t. its graph $G(f) = \{ (x,y) \in X \times Y : y = f(x) \}$

is a polyhedron in $\mathbb{R}^{n+m}$.

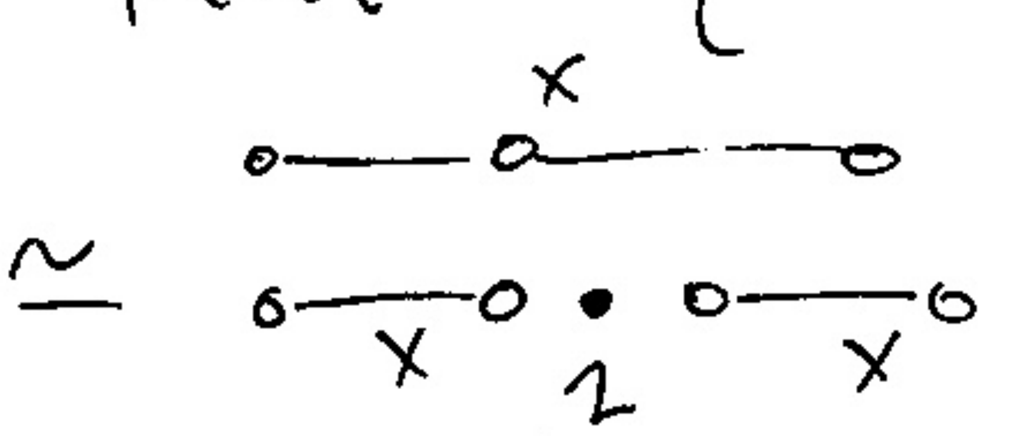
f does not have to be continuous.

$\rightsquigarrow$ category of semi-algebraic sets.

Q: What are the isomorphism classes of polyhedrons? How to classify them?

Easier question: Bounded polyhedrons.

Hint: $[\circ\!\!-\!\!\circ] =: x$. $\mathbb{Z}[x]/\sim$



If allow cancellations, then in $\mathbb{Z}$. $\rightsquigarrow$ polyhedron $\mapsto \chi$