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Let $\mathbb{F}$ be a field.

Let V vector space over $\mathbb{F}$, $\dim V = n$

$Gr(k,V) = \{$ k-subspaces of $V\}$

Choose a n-frame $(v_1, \dots, v_n)$ of V, i.e. $v_1, \dots, v_n$ linearly indep. Let

$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$, where $V_i = \text{span}\{v_1, \dots, v_i\}$

This is a flag.

$\forall X \in Gr(k,V)$, $0 \le \dim(X \cap V_0) \le \cdots \le \dim(X \cap V_n) = k$.

Given $\sigma = \{s_1, \dots, s_k\} \subseteq [n] := \{1, \dots, n\}$.
k-subset, with $s_1 < s_2 < \cdots < s_k$,

define $X_\sigma := \{X \in Gr(k,V) : \dim(X \cap V_{s_i}) = i$,
$\dim(X \cap V_{s_i - 1}) = i-1$, $i = 1, 2, \dots, k\}$.

Let $U_i = \{x_1 v_1 + \cdots + x_i v_i \in V_i : x_i \neq 0\}$.
$= V_i \setminus V_{i-1}$

This is an open set.

__Lem__ $X \in X_\sigma$ iff X has a basis of vectors $\{u_1, \dots, u_k\}$ with $u_j \in U_{s_j}$.

The basis $u_1, \dots, u_k$ can be selected s.t. the coordinate of u_i in terms of $v_1, \dots, v_n$ has the form $(x_1, \dots, x_{i-1}, 1, 0, \dots, 0)$.

So the coordinate vectors $u_1, \dots, u_k$ form a $k \times n$ matrix

$$\begin{bmatrix} * & \cdots & * & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & \cdots & * & * & * & \cdots & * & \cdots & * & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ * & \cdots & * & * & * & & & \cdots & * & 0 & * & \cdots & * & 1 & 0 \cdots & 0 \end{bmatrix} =: M$$

Every plane can be written in this way.

$$X = \mathrm{Row}(M)$$

This is a ~~Schubert cell~~ Schubert cell of dim (# of ~~stars~~)

$$Gr(k, V) = \bigsqcup_{\sigma} X_\sigma .$$

Schubert cells are open.

$\overline{X_\sigma}$ is called Schubert variety / closed Schubert cell.

If closed, then x_i possibly 0. So each $\overline{X_\sigma}$ looks like M with 1 replaced by $*$.

To see it as a variety, we need to write it as a set of equations. We will have to introduce the so-called Plücker coordinates of $Gr(k, V)$

$$Gr(k, V) \hookrightarrow \mathbb{P}^{\binom{n}{k}}$$

$$X \xmapsto{M(X)} \left(\det(M|I) \right)_{\substack{I \subset [n] \\ |I| = k}}$$

If $M' = AM$,
$\det(M'|I) = \det A \, \det(M|I)$

The sets $\{X_\sigma : \sigma\}$, $\{\overline{X_\sigma} : \sigma\}$ form posets, where

$$\overline{X_\sigma} \subseteq \overline{X_\tau} \iff \sigma \leq \tau,$$

$\left\{\begin{array}{l}\{X_\sigma : \sigma\} \\ \{\overline{X_\sigma} : \sigma\}\end{array}\right\}$ system of Schubert cells relative to the basis $(v_1, \dots, v_n)$.

Full system of Schubert cells.

$$Sch(Gr(k,V)) = \{X_{\sigma(v_1,\dots,v_n)} : (v_1,\dots,v_n)\}.$$

Consider $\mathcal{B}(Sch(Gr(k,V)))$ ← Boolean algebra

Chen's Conjecture: $\exists!$ valuation $v : \mathcal{B}(Gr(k,V)) \to \mathbb{Z}[q]$ s.t. $v(X_\sigma) = q^{\dim X_\sigma}$.