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Plücker coordinates.

Recall last ~~lec~~ lecture, we had $Gr(n, \mathbb{R}^d) \hookrightarrow \mathbb{P}\mathbb{R}^{\binom{d}{n}}$.

Let P n-plane in $\mathbb{R}^d$ (n-dim subspace), with

$$P = \text{Row}\, M_{n\times d}$$

We send it to $(\det M_I)_{I \in \binom{[d]}{n}}$, i.e. I ~~is the elem~~ is an n-subset of $[d]$, and M_I is the submatrix of M with columns ~~which are~~ whose column numbers in ~~the~~ M are in I.

$(\det M_I)$ has $\binom{d}{n}$ entries. View this as a point in $\mathbb{P}\mathbb{R}^{\binom{d}{n}}$.

Let M' be another $n\times d$ matrix with ~~A~~ $P = \text{Row}\, M'$.

Then $M' = A_{n\times n} M$ and $(\det M'_I)_I = (\det A)(\det M_I)_I$.

So $P \mapsto \mathbb{P}\mathbb{R}^{\binom{d}{n}}$ is well-defined.

Let $I = \{i_1 < \cdots < i_n\}$ and $J = \{j_1 < \cdots < j_n\}$ ordered n-tuples, in $\binom{[d]}{n}$.

We reset notation

$$Gr(n, \mathbb{R}^d) \hookrightarrow \mathbb{P}\mathbb{R}^{\binom{d}{n}}$$
$$\text{n-plane } X \mapsto P(X) \quad (P \text{ for Plücker}).$$

Fix $K = \{k_1 < \cdots < k_r\} \subseteq [n]$,

$L = \{l_1 < \cdots < l_r\} \subseteq [n]$.

Compute

$x'_{ke} = P_{i_1 \cdots \hat{i}_k l \cdots i_n}(X) = P_{i_1 \cdots \hat{i}_k l \cdots i_n}(Y) = y_{kl}$,

where $Y = AX$,

$Gr(n, \mathbb{R}^d) \hookrightarrow \mathbb{P}\mathbb{R}^{\binom{d}{n}}$ is not a surjection. We need to find polynomial equations (in homogeneous coordinates) which are the constraints. These are homogeneous polynomials called Plücker relations.

$$\det Y_J = \sum_L (-1)^{\sum_{a\in K} a + \sum_{b\in L} b} \det(Y_{K,J_L}) \det(Y_{[n]\setminus K, J_{[n]\setminus L}})$$

This is a quadratic relation.

These ~~types~~ types of equations should also describe closed Schubert cells. $\longrightarrow$ Schubert calculus.

$$P_{i_1 \cdots i_n} P_{j_1 \cdots j_n} = \sum_{k \le l_1 < \cdots < l_r \le n} P_{i_1 \cdots \hat{i}_{l_1} j_{e_1} \cdots \hat{j}_{k_r} j_{e_r} \cdots i_n}$$

$$P_{j_1 \cdots \hat{j}_{e_1} j_{e_r} \cdots \hat{j}_{e_r} i_{e_r} \cdots j_n} \quad -(*)$$

— Plücker relations.

Example Let $Gr(n, d) = Gr(n, \mathbb{F}^d)$ (of course, the above work for every $\mathbb{F}$)

$Gr(3, 6)$ # of Plücker coordinates $= \binom{6}{3} = 20$.

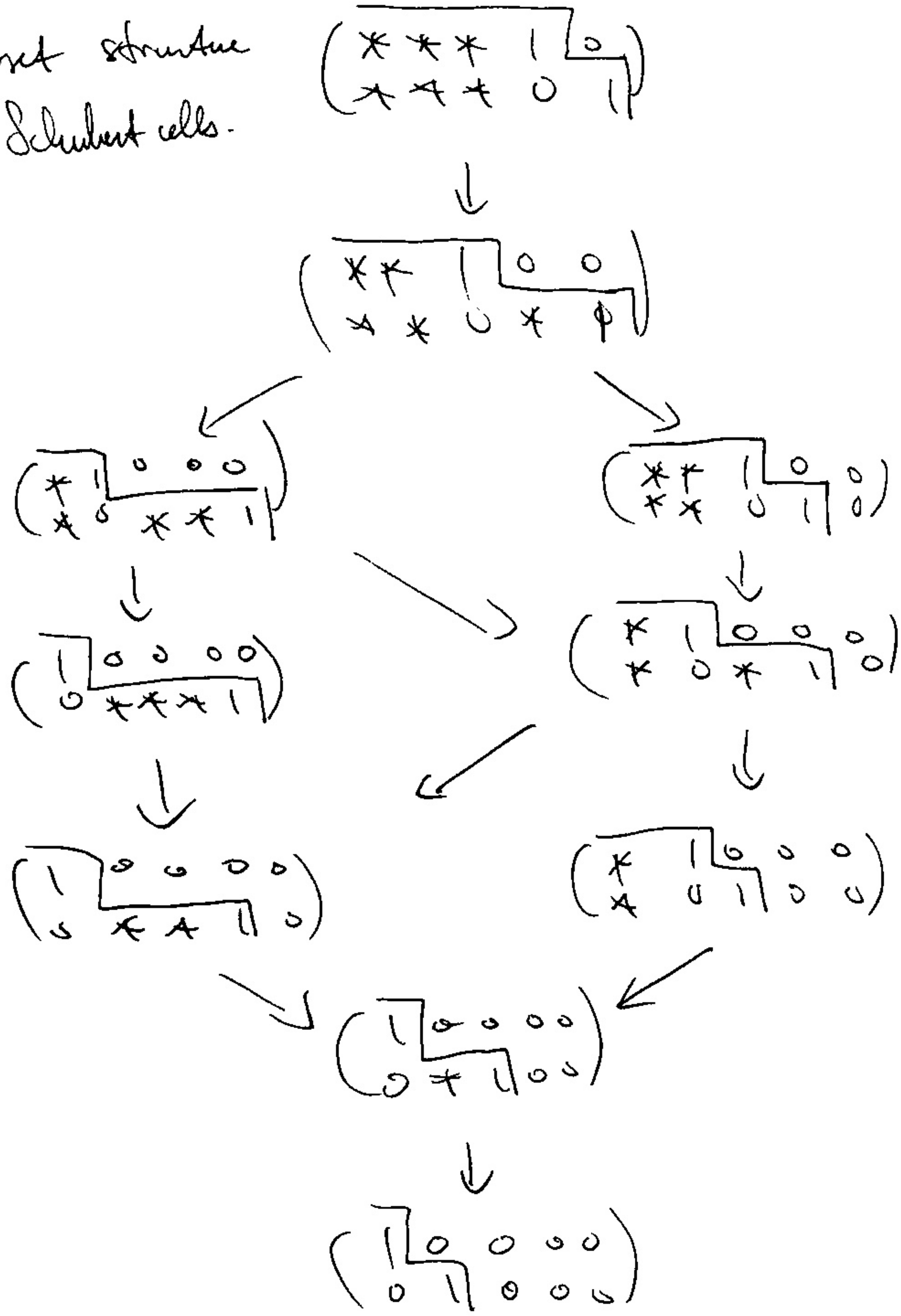
There are exactly 3 subsequences of 4, 5, 6, of length 1.

$$P_{12\underline{3}}P_{456} = P_{12\underline{4}}P_{356} + P_{12\underline{5}}P_{435} + P_{12\underline{6}}P_{45\underline{3}}.$$

For $Gr(2,5)$ $\begin{pmatrix} * & * & * & 1 & 0 \\ * & * & * & 0 & 1 \end{pmatrix}$, indep rows. This gives a 2-plane in $\mathbb{F}^5$.

We see that the dimension of $Gr(2,5) = 6$.

Poset structure of Schubert cells.



pivot positions

$$\begin{tikzcd}[column sep=small]
& 4,5 \arrow[d] & \\
& 3,5 \arrow[dl] \arrow[dr] & \\
2,5 \arrow[d] \arrow[drr] & & 3,4 \arrow[d] \\
1,5 \arrow[d] & & 2,4 \arrow[dll] \arrow[d] \\
1,4 \arrow[dr] & & 2,3 \arrow[dl] \\
& 1,3 \arrow[d, dash] & \\
& 1,2 &
\end{tikzcd}$$

partial order $\qquad \chi(Gr_{\mathbb{R}}(2,5)) = 1-1+2-2+2-1+1=2.$

corresponds to # of pivot positions at each level.

$$\binom{5}{2}_q = \frac{[5]_q!}{[2]_q!\,[3]_q!} = \frac{[5]_q[4]_q}{[2]_q[1]_q}$$

$$= \frac{(1+q+\cdots+q^4)(1+q+q^2+q^3)}{1+q} = (1+\cdots+q^4)(1+q^2)$$

$$= \frac{(q^5-1)(q^4-1)}{(q-1)^2(1+q)}$$

$\chi(Gr_{\mathbb{C}}(2,5)) = 1+1+2+2+2+1+1 = 10$. (this time is sum of the # of pivot positions)

$$\binom{5}{2}_{q=1} = \frac{5\cdot 4}{2\cdot 1} = 10.$$

Which q do we plug in for $Gr_{\mathbb{R}}(2,5)$.

We ~~see~~ see that $\binom{5}{2}_{q=-1} = 2 = \chi(Gr_{\mathbb{R}}(2,5))$.

How this works ? ~~Even cells~~

$$\begin{pmatrix} * & * & * & 1 & 0 \\ * & * & * & 0 & 1 \end{pmatrix} \sim q^6$$

$$\begin{pmatrix} * & * & 1 & 0 & 0 \\ * & * & 0 & * & 1 \end{pmatrix} \sim q^5$$

$$\begin{pmatrix} * & 1 & 0 & 0 & 0 \\ * & 0 & * & * & 1 \end{pmatrix} \quad \begin{pmatrix} * & * & 1 & 0 & 0 \\ * & * & 0 & 1 & 0 \end{pmatrix} \sim 2q^4$$

etc. Then this is where the alternating signs for $\mathbb{R}$, and same +ve signs for $\mathbb{C}$ comes from.

Let X be any topological space.

Let $H_k(X, \mathbb{R})$ $k = 0, \ldots, \dim X$ homology group.

Associated is the ~~po~~ Poincare polynomial

$$\mathrm{Poin}(X, t) = \sum_{k=0}^{\dim X} \dim H_k(X, \mathbb{R})\, t^k.$$

$$\mathrm{Poin}\left(Gr_{\mathbb{C}}(n,d), t\right) = \sum_{k=0}^{n(d-n)} \dim_{\mathbb{R}} H_k\left(Gr_{\mathbb{C}}(n,d), \mathbb{R}\right) t^{2k}.$$

$$\mathrm{Poin}\left(Gr_{\mathbb{C}}(2,5), t\right) = \cancel{t^6 + t^5 + 2t^4 + 2t^3 + 2t^2 + t + 1} \\ t^{12} + t^{10} + 2t^8 + 2t^6 + 2t^4 + t^2 + 1.$$

$$\chi\left(Gr_{\mathbb{C}}(2,5)\right) = \mathrm{Poin}\left(Gr_{\mathbb{C}}(2,5), -1\right) = 10.$$

But $\chi\left(Gr_{\mathbb{R}}(2,5)\right) = \mathrm{Poin}\left(Gr_{\mathbb{R}}(2,5), -1\right)$

$$= \left(t^6 + t^5 + 2t^4 + 2t^3 + 2t^2 + t + 1\right)\Big|_{t=-1}$$

$= 2.$

$\longrightarrow$ Define boundary operator on cell ~~pre~~ precisely

$$\left[\partial[v_0 v_1 \ldots v_n] = \sum_{k=0}^{k} (-1)^k [v_0 v_1 \ldots \hat{v}_k \ldots v_n]\right]$$

Do a project on this.