MATH 5380 Lecture 2.

Classify polyhedra — not hard.

Ex 1 (easy, but not very easy, straightforward)

Every polyhedron in $\mathbb{R}^d$ can be written as a disjoint union of relatively open convex polyhedra.

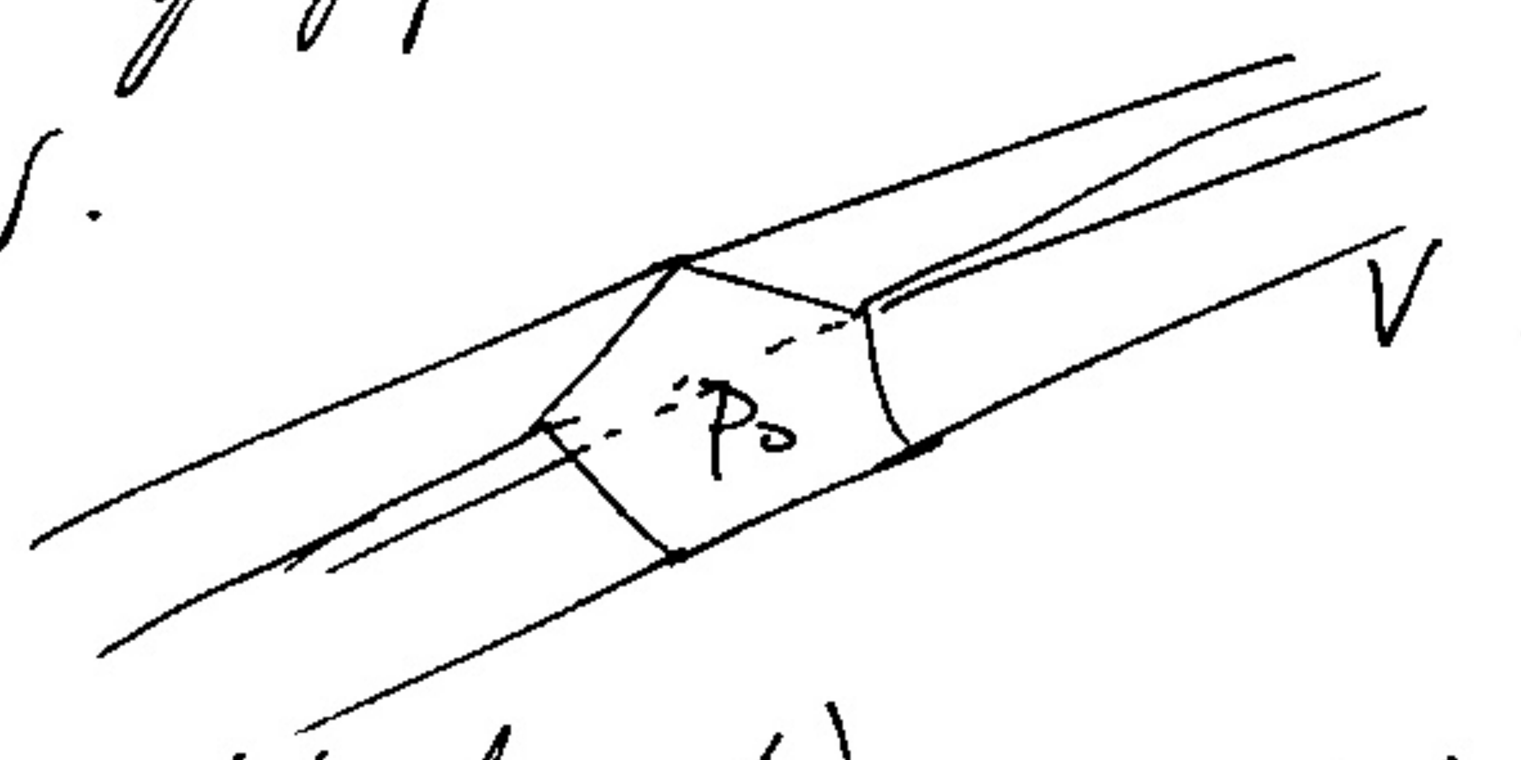
— In algebraic geometry: every algebraic variety can be blown up into smooth ones.

Ex 2 Let $P$ be an unbounded closed convex polyhedron

Then

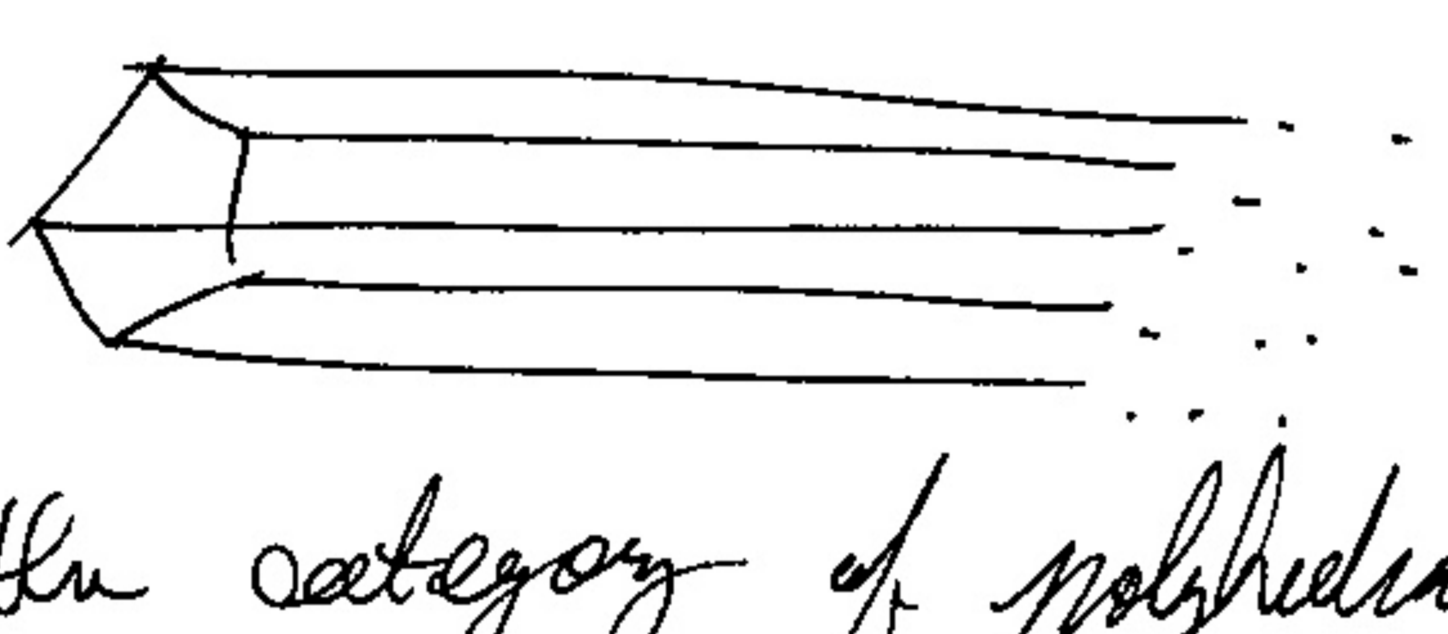
(1) $\exists$ bounded polytope $P_0$ and a ~~half line $L$~~ vector subspace $V \subseteq \mathbb{R}^d$ s.t.

$P = P_0 \oplus V$.



(2) $\exists$ (topological) polyhedral manifold $M$ and (closed) half-line $L$ s.t.

$P = M \oplus L$ (manifold w/ ~~b~~/o boundary).



Recall the category of polyhedra $\mathcal{P}$. We shall restrict our attention to ~~both~~ the category of bounded polyhedra.

$\mathcal{P}_0$ ← bounded.

$\underline{\text{Ex 3}}$ ~~Given~~ $\underset{\text{in } \mathbb{R}^n}{X} \xrightarrow{f} \underset{\text{in } \mathbb{R}^m}{Y}$ is a polyhedral map.

$\Leftrightarrow$ $f$ is a piecewise affine linear map (continuity not needed).

(i.e. $X$ can ~~be~~ be decomposed into a disjoint union of relatively open polytopes $P$ s.t. the restriction

$f|_P : P \to Y$ ~~can~~ can be extended to an affine linear map $f|_P : \langle P \rangle \to \mathbb{R}^m$, where

$\langle P \rangle$ ~~is the~~ $=$ aff sp $(P)$ the affine span of $P$, the smallest plane containing $P$.

We want to build a group.

$\forall P \in \mathbb{P}_0$, let $[P]$ — isomorphism class of $P$.

Consider the collection of finite formal linear combinations

$$\left\{ \sum_i a_i [P_i] : a_i \in \mathbb{Z}, P_i \in \mathbb{P}_0 \right\} =: F(\mathbb{P}_0) -$$

free (abelian) group, subject to the relation

$$[X \cup Y] = [X] + [Y] - [X \cap Y].$$

[Simpler category, $\underline{\underline{\text{Set}}}_{\text{finite}} =: \mathbb{S}$. We can also do this kind of thing.]

$F(\mathbb{S})/\sim$ (inclusion - exclusion) $\underset{\text{AbGrp}}{\cong} \mathbb{Z}$. (abelian group)

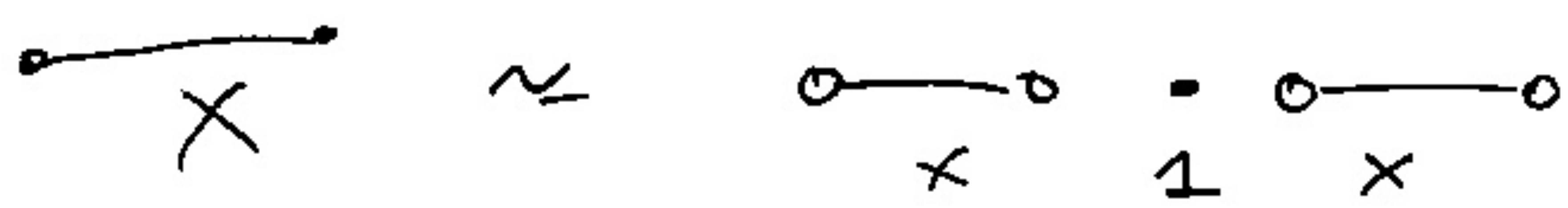
We want to lift this to a ring isomorphism.

~~$[P]\cdot[Q]$~~

$[X]\cdot[Y] := [X\times Y]$ (bilinearly).

$$\Big(\sum_i a_i[X_i]\Big)\Big(\sum_j b_j[Y_j]\Big) := \sum_{ij} a_i b_j\,[X_i\times Y_j].$$

For every category we can do this.

Let 

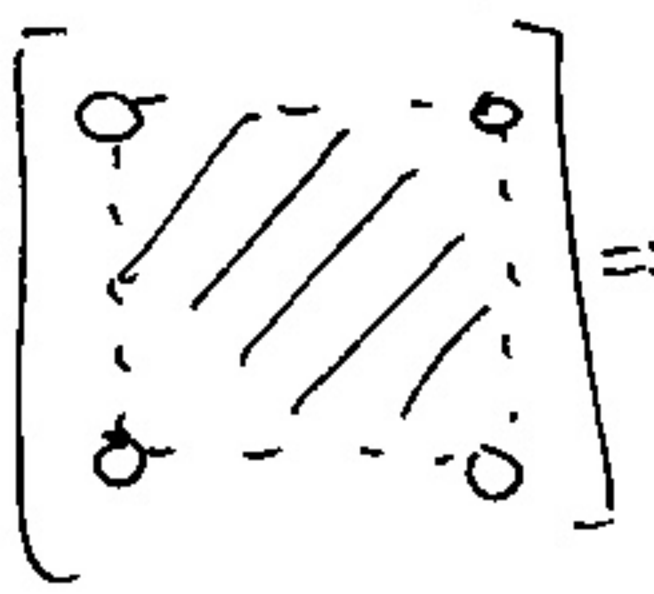
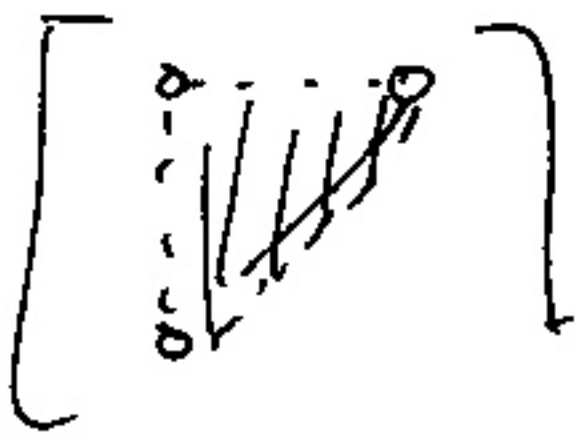

$x = 2x+1$

$x+1=0.$

$x = [\circ\!-\!\circ]$

$[P] = \sum a_i[\sigma_i^\circ]$. $\sigma_i^\circ$ – open simplices.

(Corollary of Ex 1)

 $= x^2$  $= x^2$.

$[\sigma_i^\circ] = x^{\dim \sigma_i^\circ}$.

Anyway, $F(\mathcal{P}_0) \cong \mathbb{Z}$.

$[P] \mapsto \chi(P)$.

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What if we restrict the set of morphisms to distance-preserving ones? ~~piecewise~~ localized rigid motion transformation. As ~, we can cut X into pieces and $f : X \to Y$ preserves distances on pieces but not necessarily globally.

Recall convex body := compact convex set.

The space of convex bodies, equipped with Hausdorff metric has a dense subspace being the space of polyhedra.

<u>Some phenomena in polyhedra cannot be found in the smooth case.</u>

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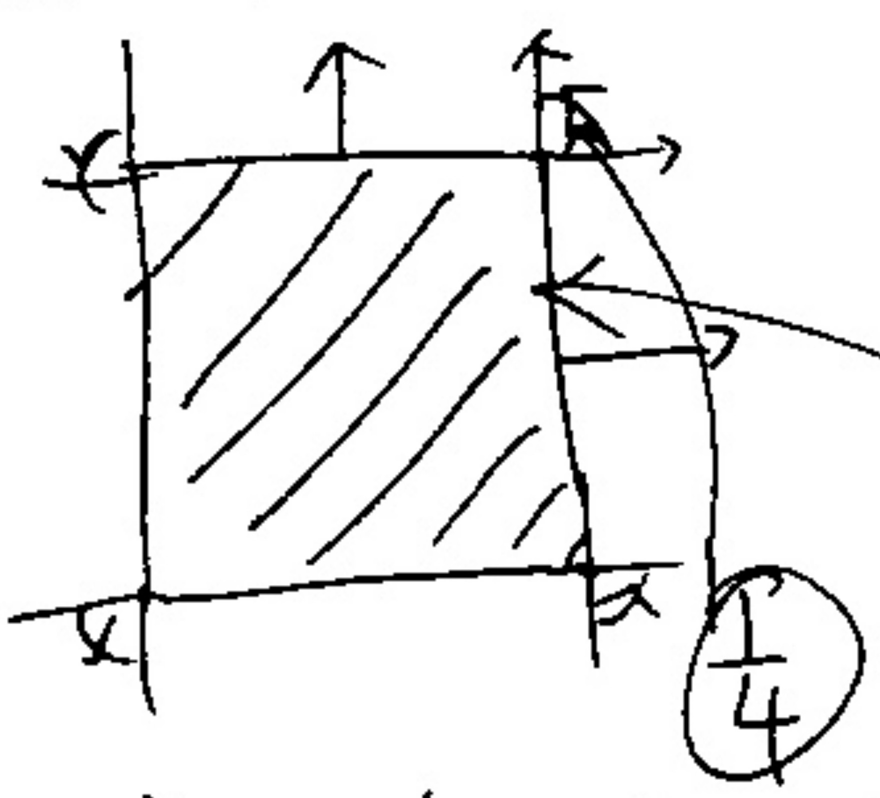
Gauss - Bonnet

Gaussian curvature for polyhedrals.

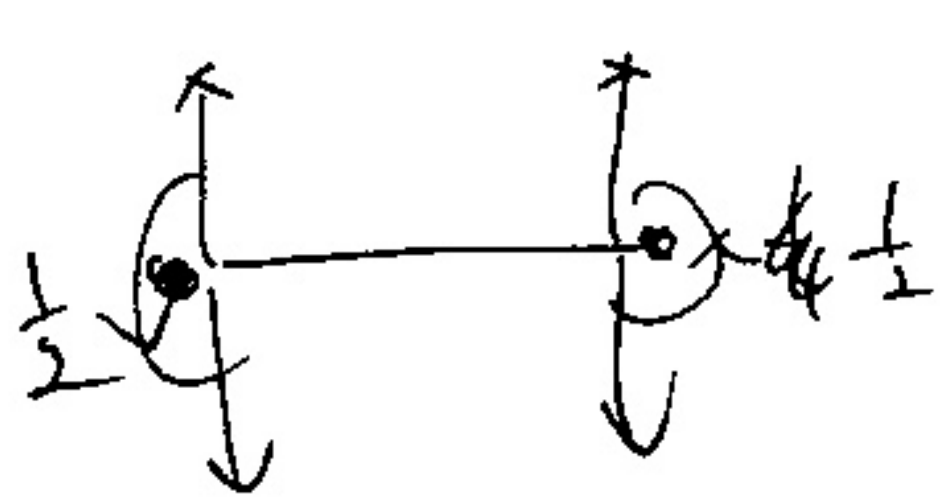



Sudden change. curvature 0 a.e.

$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = 1$.

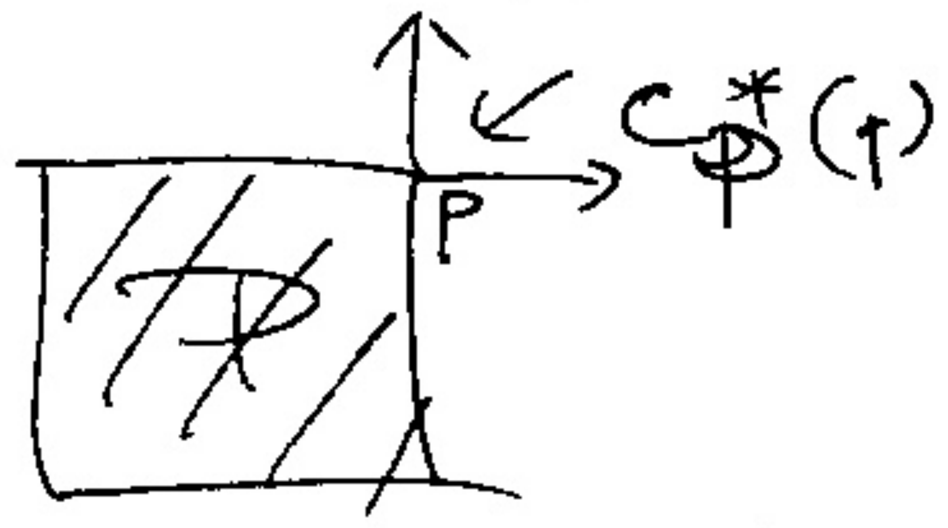
Usually on manifolds, we have tangent space. Here, we have tangent cone. — The normal cone, measures how that place is bent. The number of the cone is the proportion in a ~~disc~~ ball.

This number is the local Gaussian curvature.




$\frac{1}{2}+\frac{1}{2} = 1$.

We cease to consider members of the cones. Consider the entire cone




Consider $\mathbb{1}_{C_P^*(p)}$.

Cones can be closed and open.

tangent cones dual with normal cones.

Let F a face of P.

Define tangent cone $C(P, F)$, and $(-1)^{\dim F} \mathbb{1}_{C(P,F)}$.

If you add all faces, this function is zero a.e. ess. Only at origin is it nonzero.

That number is ~~$(-1)^{\dim}$~~ $\chi(P)$.

I'm lost, but this is related to Gauss Bonnet.

{space of cones}, idempotent automorphism on this space.

$\Rightarrow$ duality.

The set of cones form a category, with morphisms preserve linear functions.