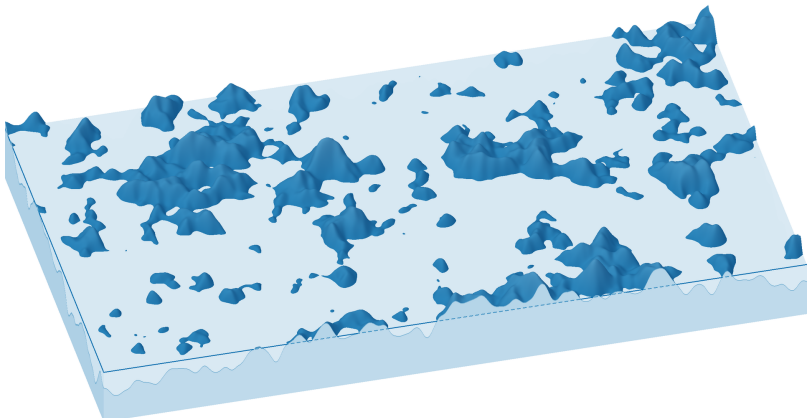
# Connectivity in Random Landscapes

**Baining Zhang**
The Hong Kong University of Science and Technology
Joint work with Dr. Stephen Muirhead (Monash University)

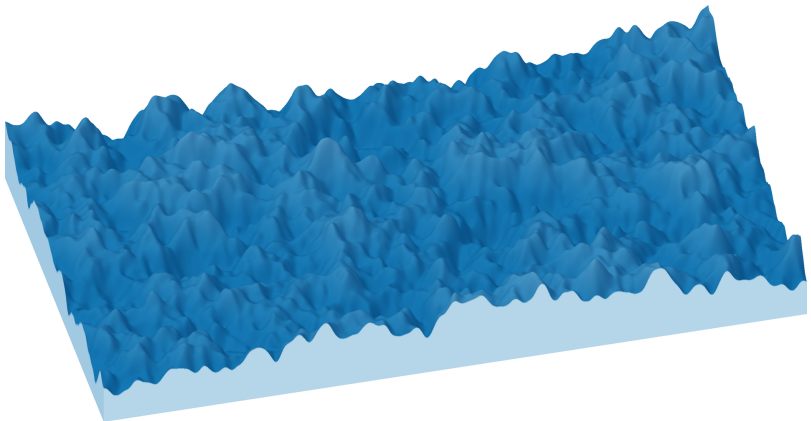




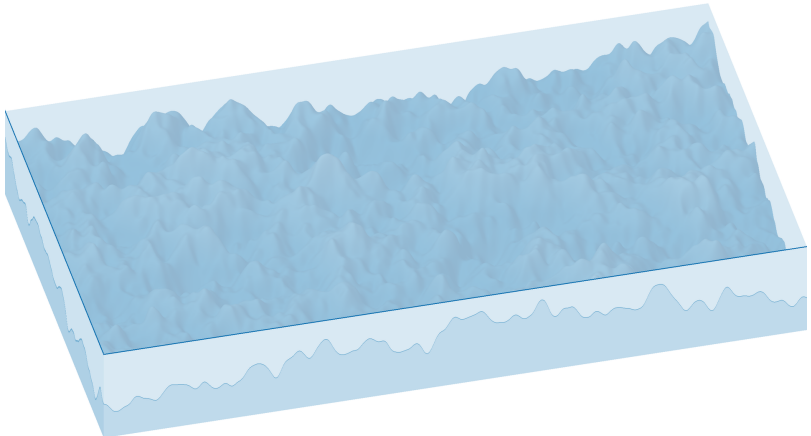
# A random landscape



# The terrain

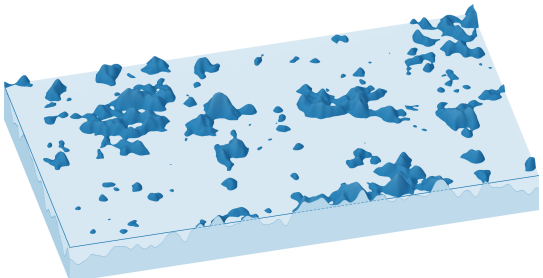


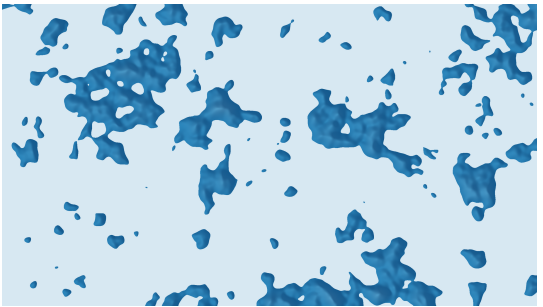
# Submerged terrain



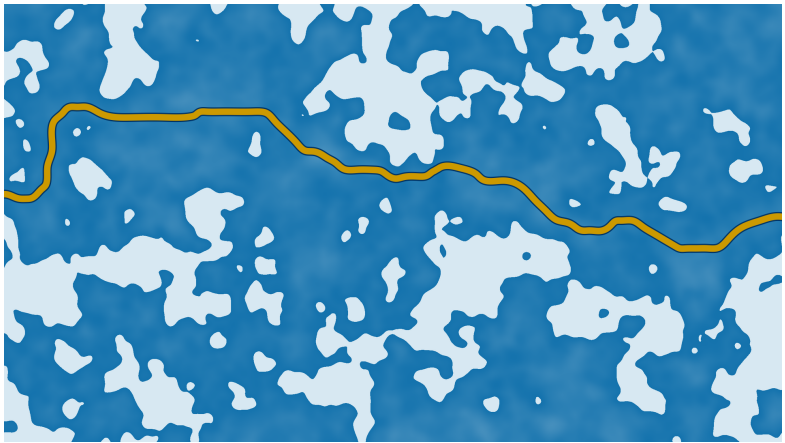
# High sea level

Mostly isolated islands emerge from the sea. As the sea level is lowered, these islands increase in size and connect with each other.





# Lower sea level



# Gaussian fields

We model the terrain with a random height function, given by a planar Gaussian field $f$, with distribution characterised by a covariance kernel $\kappa$:

$$f : \mathbb{R}^2 \to \mathbb{R} \quad \text{centred, stationary, Gaussian,} \qquad \kappa(x) = \mathbb{E}[f(0)f(x)].$$

**Assumptions (MRV23, Assumption 1.1)**

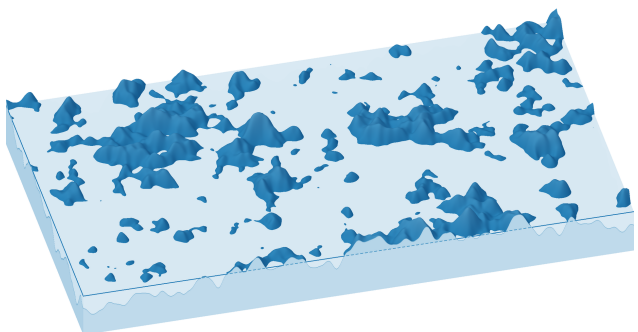
- $f$ is almost surely $C^3$.
- For every $x \neq 0$, $(f(0), f(x), \nabla f(0), \nabla f(x))$ is nondegenerate.
- $\kappa(x) \to 0$ as $|x| \to \infty$.

S. Muirhead, A. Rivera, and H. Vanneuville. *The phase transition for planar Gaussian percolation models without FKG*. Ann. Probab., 51(5):1785–1829, 2023. With an appendix by L. Köhler-Schindler.

# The Cauchy field

For concreteness, we will consider the following Gaussian field $f$ with covariance

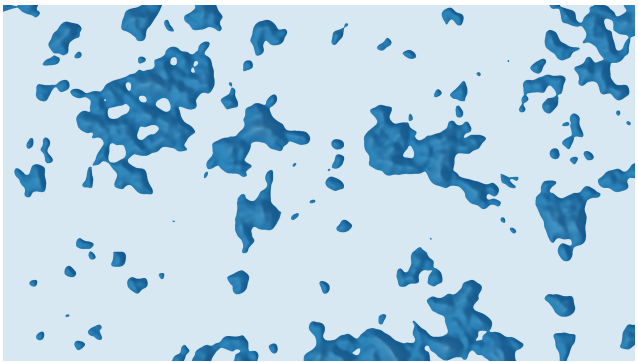
$$\kappa(x) = (1 + |x|^2)^{-1/4} \sim |x|^{-1/2}.$$



# Excursion sets

Excursion sets of $f$ correspond to exposed land above a level $-\ell$.

$$\{f \geq -\ell\} = \{x \in \mathbb{R}^2 : f(x) \geq -\ell\}$$



# Level sets

Level sets $\{f = -\ell\}$ of $f$ comprises curves in $\mathbb{R}^2$, corresponding to coastlines.



$\{f = 0\}$

# Clusters

We call connected components of excursion/level sets *clusters*. The largest cluster is highlighted.



# Percolation

We say that *percolation* occurs when the excursion set contains an infinite (unbounded) cluster. This is an infinite-volume statement.



At which levels $\ell$ does percolation occur?

# Phase transition

**Theorem (MRV23, Theorem 1.3)**

Let $f$ satisfy the Assumptions.

- If $f$ is $D_4$-symmetric, then for every $\ell < 0$, almost surely every connected component of $\{f \geq -\ell\}$ is bounded.
- If $f$ is isotropic and, for some $\delta > 0$,

$$|\kappa(x)|(\log\log|x|)^{2+\delta} \longrightarrow 0 \qquad (|x| \to \infty),$$

then for every $\ell > 0$, almost surely $\{f \geq -\ell\}$ has a unique unbounded connected component.

The critical level is zero. This is due to planarity and not simply because $f$ has zero-average.

$$\ell_c = \inf\{\ell : \mathbb{P}(\{f \geq -\ell\} \text{ percolates}) > 0\} = 0$$

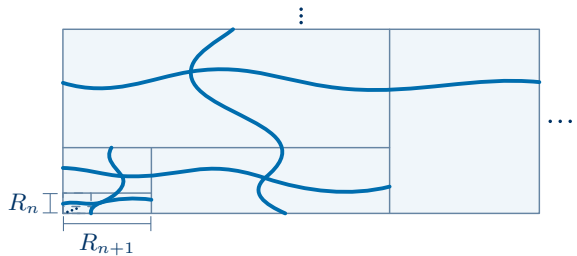
# Rectangle crossings

Infinite clusters are built from crossings.



$\mathrm{Cross}_\ell(a,b) = \{\{f \geq -\ell\}$ crosses $[0,a] \times [0,b]$ from left to right$\}$.

# From crossings to percolation



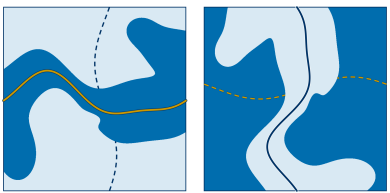

In the supercritical regime ($\ell > 0$), there exists a sequence of scales $R_{n+1} \geq 2R_n$ such that

$$\sum_n \mathbb{P}(\text{no crossing for a } R_n \times R_{n+1} \text{ rectangle}) < \infty.$$

Borel-Cantelli yields the almost sure existence of all but finitely many of these crossings, which intersect to form an infinite cluster. Hence, this picture can be observed at sufficiently large scale $R_n$.

# Planar duality and crossing probabilities

Left-to-right crossings in $\{f \geq -\ell\}$ block top-to-bottom crossings in $\{f \leq -\ell\}$.



Since $f$ is centred, $\{f \leq -\ell\} \overset{d}{=} \{f \geq -(-\ell)\}$.

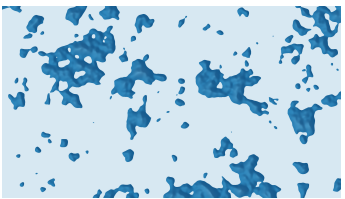
$$1 - \mathbb{P}[\mathrm{Cross}_\ell(R, aR)] = \mathbb{P}[\mathrm{Cross}_{-\ell}(aR, R)].$$

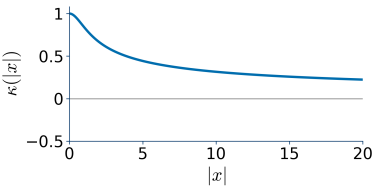
This symmetry gives us $\ell_c = 0$.

In the subcritical regime $(\ell < 0)$, as the scale $R$ of the rectangles increase, how quickly does the crossing probability decay?

# Cauchy field with $\alpha = \frac{1}{2}$

$$\kappa(x) = (1 + |x|^2)^{-1/4} \sim |x|^{-1/2}$$






**Corollary of MS24, Theorem 1.11(3)**

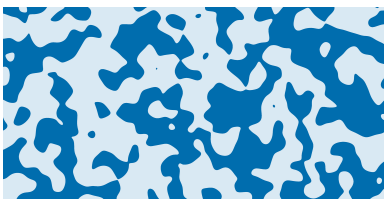
For each fixed $\ell < 0$ and $a > 0$, there exist $c, C > 0$ such that

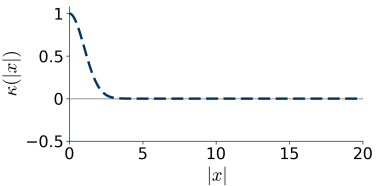
$$\mathbb{P}[\mathrm{Cross}_\ell(R, aR)] \leq Ce^{-c\sqrt{R}} \qquad \forall R \geq 1.$$


S. Muirhead and F. Severo. *Percolation of strongly correlated Gaussian fields I. Decay of subcritical connection probabilities*. Probab. Math. Phys., 5(2):357–412, 2024.

# Bargmann–Fock field

$$\kappa(x) = e^{-|x|^2/2}$$






**MV20, Theorem 1.11**

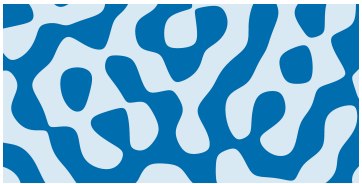
For each fixed $\ell < 0$ and $a > 0$, there exist $c, C > 0$ such that

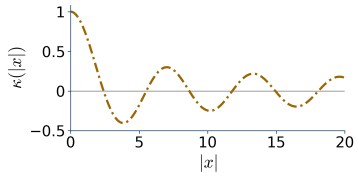
$$\mathbb{P}[\mathrm{Cross}_\ell(R, aR)] \leq Ce^{-cR} \qquad \forall R \geq 1.$$


S. Muirhead and H. Vanneuville. *The sharp phase transition for level set percolation of smooth planar Gaussian fields*. Ann. Inst. H. Poincaré Probab. Statist., 56(2):1358–1390, 2020.

# Random planar wave

$$\kappa(x) = J_0(|x|) = \sqrt{\frac{2}{\pi|x|}}\cos\left(|x| - \frac{\pi}{4}\right) + O(|x|^{-3/2}) \qquad (|x| \to \infty).$$






**MRV23, Theorem 1.7**

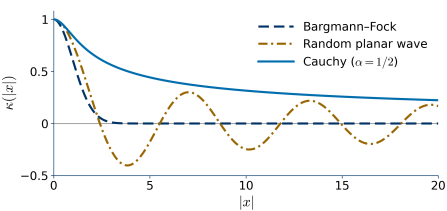
For each fixed $\ell < 0$ and $a > 0$, there exist $c, C > 0$ such that

$$\mathbb{P}[\mathrm{Cross}_\ell(R, aR)] \le Ce^{-c\sqrt{\log R}} \qquad \forall R \ge 1.$$

$\mathbb{P}[\mathrm{Cross}_\ell(R, aR)] \le Ce^{-cR}$ (conjectured, MRV23, §1.3 and Question 1.14).

# Sharpness and universality

Decay rates in crossing probabilities quantify the *sharpness* of the percolation phase transition.




**Upper bounds on $\mathbb{P}[\mathrm{Cross}_\ell(R, aR)]$**

At fixed $\ell < 0$ and $a > 0$

| | |
|---|---|
| Cauchy | $Ce^{-c\sqrt{R}}$ |
| BF | $Ce^{-cR}$ |
| RPW | $Ce^{-cR}$ (conjectured) |

**Conjectured universality classes**

| | |
|---|---|
| BF, RPW | Bernoulli percolation |
| Cauchy | A strongly correlated class |

# The zero level lines



**Theorem/Conjecture**

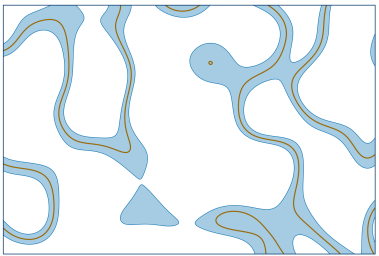
Zero level lines $\{f = 0\}$ are bounded almost surely.

MV20, Theorem 1.7(i); Ale96, Theorem 3.4.
For the random plane wave, boundedness is conjectured but expected (MRV23, Conjecture 1.11).

K. S. Alexander. *Boundedness of level lines for two-dimensional random fields*. Ann. Probab., 24(4):1653–1674, 1996.

# Level thickening

We can thicken $\{f = 0\}$ by allowing a tolerance for the height of $f$.



Almost surely, $\{|f| \leq \varepsilon\}$ has a unique unbounded connected component for every $\varepsilon > 0$.

MV20, Theorem 1.7; Corollary of MRV23, Theorem 1.7.

Thickening by level is non-uniform. The width scales with $|\nabla f|^{-1}$.

# References


MRV23 S. Muirhead, A. Rivera, and H. Vanneuville. *The phase transition for planar Gaussian percolation models without FKG*. Ann. Probab., 51(5):1785–1829, 2023. With an appendix by L. Köhler-Schindler.

MS24 S. Muirhead and F. Severo. *Percolation of strongly correlated Gaussian fields I. Decay of subcritical connection probabilities*. Probab. Math. Phys., 5(2):357–412, 2024.

MV20 S. Muirhead and H. Vanneuville. *The sharp phase transition for level set percolation of smooth planar Gaussian fields*. Ann. Inst. H. Poincaré Probab. Statist., 56(2):1358–1390, 2020.

Ale96 K. S. Alexander. *Boundedness of level lines for two-dimensional random fields*. Ann. Probab., 24(4):1653–1674, 1996.


## Usage of AI tools

AI tools were used in formatting and drawing figures. All textual exposition is human.