The Quenched Critical Point of Self-Avoiding Walk on Infinite Graphs with Random Conductances

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Abstract

We study the quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}(x)$ of the self-avoiding walk among random conductances, generalizing the results of Chino and Sakai, (J Stat Phys (2016) 163:754–764) to the case of infinite connected graphs of bounded degrees with i.i.d. conductances. We show that the connective constant and the quenched critical point are independent of the choice of the reference point. We also show that the quenched critical point is non-random and satisfies the inequality $h_0 - \beta\,\mathbf{E}\,[X_e] \leq \hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}(x)$ almost surely.

1 Introduction

The self-avoiding walk is a model of statistical mechanics introduced by Flory [Flo53] for modelling polymer chains. It has since then become a very interesting object in mathematics, leading to many rigorous results and even more open problems, most of which are supported by numerical simulations and physical considerations. We refer to [MS93] for a thorough introduction to this subject.

It is more realistic to model a polymer chain in a random, inhomogeneous medium. Recently, the behaviour of self-avoiding walks has been studied on infinite supercritical percolation clusters on $\mathbb{Z}^d$ in [Lac14b; Lac14a] by Lacoin. In dimension two, it was established that the quenched connective constant (and hence the quenched critical point) is strictly less than the annealed above the critical probability. For sufficiently high dimensions, it was shown that there exists a regime where this strict inequality holds. This suggests a strong disorder for the self-avoiding walk, which indicates that the disorder from percolation alters the behaviour of the trajectories.

Inspired by these results, Chino and Sakai studied the behaviour of self-avoiding walks on ergodic and translation-invariant random conductances on $\mathbb{Z}^d$ in [CS16]. The quenched critical point was shown to be a constant on a set of probability one, and does not depend on the point of reference. Furthermore, upper and lower bounds on the quenched critical points were provided.

In this report, we generalise some results in [CS16] to i.i.d. random conductances on infinite connected graphs with bounded degree.

1.1 Definitions and Set-up

Let $\mathcal{C}$ be an infinite connected graph with bounded degree. A specific class of examples of such graphs is given by infinite supercritical (Bernoulli) percolation clusters on $\mathbb{Z}^d$, where $\mathcal{C}$ is the unique infinite cluster in the random subgraph of $\mathbb{Z}^d$ obtained by removing edges in an i.i.d. fashion with probability $1-p$, for $p > p_c$, where p_c is the percolation threshold, i.e. the probability above which a unique infinite cluster exists almost surely.

We first introduce the self-avoiding walk (SAW). A (nearest-neighbour) path $\omega = (\omega_1, \ldots, \omega_n)$ is a SAW if its vertices do not repeat, i.e. $\omega_i \neq \omega_j$ for $i \neq j$. Let $\Omega(x, y)$ denote the set of SAWs from x to y in $\mathcal{C}$. Let $\Omega(x) = \bigcup_{y \in \mathcal{C}} \Omega(x, y)$ and $\Omega(x; n) = \{\omega \in \Omega(x) \,;\, |\omega| = n\}$, where for a given SAW ω, we denote its length by $|\omega|$.

We view each edge of the path as a bond between consecutive monomers, which incurs an energy cost of some $h \in \mathbb{R}$. We define the susceptibility at $x \in \mathcal{C}$ as

$$\chi_h(x) = \sum_{\omega \in \Omega(x)} e^{-h|\omega|}.$$

The convergence of this series is the main issue of interest in this report. Note that $\chi_h(x)$ is decreasing in h. Hence, we can define the critical point $h_0(x)$ for the susceptibility:

$$h_0(x) = \inf \{h \in \mathbb{R} : \chi_h(x) < \infty\}.$$

We define two other fundamental quantities for the SAW: the number of n-step SAWs and the two-point function

$$c(x;n) = \sum_{\omega \in \Omega(x)} \mathbf{1}_{\{|\omega|=n\}} \quad \text{and} \quad G_h(x,y) = \sum_{\omega \in \Omega(x,y)} e^{-h|\omega|}.$$

We have the following important relation:

$$\chi_h(x) = \sum_{n=0}^{\infty} c(x;n)e^{-hn} = \sum_{y \in \mathcal{C}} G_h(x,y).$$

In other words, the susceptibility can be viewed as a generating function for $c(x;n)$.

Proposition 1. The connective constant $\mu(x)$ at $x \in \mathcal{C}$, given by

$$\mu(x) = \limsup_{n \to \infty} c(x;n)^{\frac{1}{n}},$$

exists and is a constant independent of x. We write $\mu \equiv \mu(x)$.

The proof of this proposition will be given in the next section.

The connective constant μ is a measure of the growth of the number of n-step SAWs in the graph. It also gives the critical point of the susceptibility.

Proposition 2. The critical point $h_0(x)$ is constant in x and is given by

$$h_0 = \log \mu.$$

Proof. Let $h < \log \mu$. Then there exists $\varepsilon > 0$ such that $h < \log(\mu - \varepsilon)$. Since $\mu = \limsup_{n \to +\infty} c(x;n)^{\frac{1}{n}}$, there exists an unbounded and increasing sequence $(n_k)_{k \in \mathbb{N}}$ of integers such that

$$c(x;n_k) \geq (\mu - \varepsilon)^{n_k}.$$

Hence, we have

$$\begin{aligned}
\chi_h(x) &= \sum_{n=0}^{+\infty} c(x;n)e^{-hn}, \\
&\geq \sum_{k=0}^{+\infty} c(x;n_k) \cdot \frac{1}{(\mu-\varepsilon)^{n_k}}, \\
&\geq \sum_{k=0}^{+\infty} \frac{(\mu-\varepsilon)^{n_k}}{(\mu-\varepsilon)^{n_k}} = \sum_{k=0}^{+\infty} 1 = +\infty.
\end{aligned}$$

For the other direction, let $h > \log \mu$. Then there exists $\varepsilon > 0$ such that $h > \log(\mu + \varepsilon)$. For this ε, there exists N such that for all $n \geq N$, we have $c(x; n) \leq (\mu + \varepsilon)^n$. Hence,

$$\begin{aligned}
\chi_h(x) &= \sum_{n=0}^{+\infty} c(x;n)e^{-hn}, \\
&\leq \sum_{n=0}^{N-1} c(x;n)e^{-hn} + \sum_{n=N}^{+\infty} (\mu+\varepsilon)^n e^{-hn}, \\
&\leq \sum_{n=0}^{N-1} c(x;n)e^{-hn} + \sum_{n=N}^{+\infty} \left[e^{-(h-\log(\mu+\varepsilon))}\right]^n, \\
&< +\infty.
\end{aligned}$$

□

Note that although the critical point does not depend on x, the (finite) value of the susceptibility $\chi_h(x)$ still does.

We now place random conductances on the edges. Let $\mathbb{E}$ denote the set of edges with both endpoints in $\mathcal{C}$ and $\mathbf{X} = \{X_e\}_{e\in\mathbb{E}}$ be a collection of independent and identically distributed random variables with finite expectation (independent of the realization of $\mathcal{C}$). Intuitively, X_e represents the conductance on the edge e.

Given a random environment $\mathbf{X}$ and a strength of randomness $\beta \geq 0$, we define the quenched susceptibility at $x \in \mathcal{C}$ by

$$\hat{\chi}_{h,\beta,\mathbf{X}}(x) = \sum_{\omega\in\Omega(x)} \exp\left\{-\sum_{j=1}^{|\omega|}(h + \beta X_{e_j})\right\},$$

where $e_j = (\omega_{j-1}, \omega_j)$. Similarly, we define the other two observables:

$$\hat{c}_{\beta,\mathbf{X}}(x;n) = \sum_{\omega\in\Omega(x)} \mathbf{1}_{\{|\omega|=n\}} \exp\left\{-\sum_{j=1}^{n} \beta X_{e_j}\right\}, \text{ and}$$

$$\hat{G}_{h,\beta,\mathbf{X}}(x,y) = \sum_{\omega\in\Omega(x,y)} \exp\left\{-\sum_{j=1}^{|\omega|}(h + \beta X_{e_j})\right\}.$$

They are related to the quenched susceptibility by

$$\hat{\chi}_{h,\beta,\mathbf{X}}(x) = \sum_{n=0}^{\infty} \hat{c}_{\beta,\mathbf{X}}(x;n)e^{-hn} = \sum_{y\in\mathcal{C}} \hat{G}_{h,\beta,\mathbf{X}}(x,y).$$

The quenched critical point is defined as

$$\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}(x) = \inf\left\{h \in \mathbb{R} : \hat{\chi}_{h,\beta,\mathbf{X}}(x) < \infty\right\}.$$

We also define the annealed critical point, given by

$$\hat{h}^{\mathsf{a}}_{\beta}(x) = \inf\left\{h \in \mathbb{R} : \mathbf{E}\left[\hat{\chi}_{h,\beta,\mathbf{X}}(x)\right] < \infty\right\}.$$

If $\mathbf{E}[\hat{\chi}_{h,\beta,\mathbf{X}}(x)] < \infty$, it must be the case that $\hat{\chi}_{h,\beta,\mathbf{X}}(x) < \infty$ on a set of probability one. Hence, we see that $\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}(x) \le \hat{h}^{\mathsf{a}}_{\beta}(x)$ almost surely.

Assuming that the moment generating function of X_e exists, we can compute the annealed critical point.

Recall that $(X(e))_{e\in\mathbb{E}}$ is a collection of i.i.d. random variables. Writing $\lambda_\beta = \mathbf{E}\left[e^{-\beta X_e}\right]$, we have

$$\begin{aligned}
\mathbf{E}\left[\hat{c}_{\beta,\mathbf{X}}(x,n)\right] &= \sum_{\omega\in\Omega(x)} \mathbf{1}_{\{|\omega|=n\}}\, \mathbf{E}\left[\exp\left\{-\beta\sum_{j=1}^{n} X_{e_j}\right\}\right], \\
&= \sum_{\omega\in\Omega(x)} \mathbf{1}_{\{|\omega|=n\}} \prod_{j=1}^{n} \mathbf{E}\left[e^{-\beta X_{e_j}}\right], \\
&= \lambda_\beta^n\, c(x;n).
\end{aligned}$$

Hence, the annealed susceptibility is given by

$$\begin{aligned}
\mathbf{E}\left[\hat{\chi}_{h,\beta,\mathbf{X}}(x)\right] &= \sum_{n=0}^{\infty} c(x;n)\lambda_\beta^n e^{-hn}, \\
&= \sum_{n=0}^{\infty} c(x;n) e^{-(h-\log\lambda_\beta)n}, \\
&= \chi_{h-\log\lambda_\beta}(x).
\end{aligned}$$

Therefore, we obtain the annealed critical point

$$\hat{h}^{\mathsf{a}}_{\beta} = h_0 + \log\lambda_\beta.$$

By Jensen's inequality, we have

$$\log\lambda_\beta = \log\mathbf{E}\left[e^{-\beta X_e}\right] \ge -\beta\,\mathbf{E}\left[X_e\right],$$

which implies that

$$\hat{h}^{\mathsf{a}}_{\beta} \geq h_0 - \beta\, \mathbf{E}\left[X_e\right].$$

This leads us to the main theorem, whose proof is given in the last section.

Theorem 1. Let $\beta \geq 0$. The quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}(x)$ is almost surely constant in x and satisfies the following inequality:

$$h_0 - \beta\, \mathbf{E}\left[X_e\right] \leq \hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}(x) \leq \hat{h}^{\mathsf{a}}_{\beta} \quad \text{almost surely.}$$

2 The Connective Constant

In this section, we show that the connective constant $\mu(x)$ does not depend on the point of reference x.

Lemma 1 (Lemma 2.1, [Lac14b])**.** For an infinite connected graph $\mathcal{C}$ with bounded degree, the connective constant $\mu = \limsup_{n\to+\infty} c(x;n)^{\frac{1}{n}}$ exists and does not depend on the choice of $x \in \mathcal{C}$.

Proof. Let D denote the maximum degree of the graph $\mathcal{C}$. Then for any $x \in \mathcal{C}$, we have

$$c(x;n) \leq D(D-1)^{n-1}.$$

Therefore,

$$\limsup_{n\to+\infty} c(x;n)^{\frac{1}{n}} \leq D-1.$$

This shows that the connective constant exists.

Since $\mathcal{C}$ is connected, it suffices to show that for any adjacent vertices $u, v \in \mathcal{C}$, we have

$$\limsup_{n\to+\infty} c(u;n)^{\frac{1}{n}} = \limsup_{n\to+\infty} c(v;n)^{\frac{1}{n}}.$$

Let $\Omega_1(u;n)$ denote the set of n-step SAWs starting at u which do not pass through v and $\Omega_2(u,v;n)$ denote the set of n-step SAWs starting at u and ending at v. Then we have the following relations:

$$|\Omega_2(u,v;n)| = |\Omega_2(v,u;n)| \leq c(v;n),$$

$$|\Omega_1(u;n)| \leq c(v;n+1).$$

The first line follows from symmetry and inclusion. For the second line, notice that for any $\omega \in \Omega_1(u;n)$, we can extend it to a $(n+1)$-step SAW by adding the edge (v,u) at the start of the path, which injects $\Omega_1(u;n)$ into $\Omega(v;n+1)$.

Given a SAW ω, let k denote its time of visit to v. We can decompose the set $\Omega(u;n)$ according to k:

$$\begin{aligned} c(u;n) = |\Omega(u;n)| &\leq \sum_{k=1}^{n} |\Omega_2(u,v;k)||\Omega(v,n-k)| + \underbrace{|\Omega_1(u;n)|}_{k=\infty,\text{ i.e. does not visit } v} \\ &\leq \sum_{k=1}^{n} c(v;k)c(v;n-k) + c(v;n+1). \end{aligned}$$

Notice that the first sum is a convolution. We are going to bound it.

Fix $\varepsilon > 0$. Then there exists N such that for any $n \geq N$, $c(v;n) \leq (\mu(v)+\varepsilon)^n$. In the following, let $n > 2N$. We will split the sum into three parts.

Part 1: $1 \leq k \leq N$ Let $C = c(v;N)$. Then $C \geq c(v,k)$ for any $k \leq N$. Moreover, we have $n-k > N$, so $c(v;n-k) \leq (\mu(v)+\varepsilon)^{n-k}$. Hence,

$$\sum_{k=1}^{N} c(v;k)c(v;n-k) \leq NC(\mu(v)+\varepsilon)^n.$$

Part 2: $N+1 \leq k \leq n-N-1$ For $k > N$, we have $c(v;k) \leq (\mu(v)+\varepsilon)^k$ and $c(v;n-k) \leq (\mu(v)+\varepsilon)^{n-k}$, which gives the bound

$$\sum_{k=N+1}^{n-N-1} c(v;k)c(v;n-k) \leq \sum_{k=N+1}^{n-N-1} (\mu(v)+\varepsilon)^k(\mu(v)+\varepsilon)^{n-k} = (n-2N-1)(\mu(v)+\varepsilon)^n.$$

Part 3: $n-N \leq k \leq n$ For $k \geq n-N > N$, we have $c(v;k) \leq (\mu(v)+\varepsilon)^k$ and $c(v;n-k) \leq C = c(v;N)$. Therefore,

$$\sum_{k=n-N}^{n} c(v;k)c(v;n-k) \leq C(\mu(v)+\varepsilon)^n.$$

We put the above together. Let D denote the maximum degree of the graph $\mathcal{C}$. For $n > 2N$, we have

$$c(u;n) \leq \left(NC + (n-2N-1) + C\right)(\mu(v)+\varepsilon)^n + c(v;n+1),$$

$$\leq (NC + (n - 2N - 1) + C + D)\,(\mu(v) + \varepsilon)^n,$$

$$\limsup_{n\to+\infty} c(u;n)^{\frac{1}{n}} \leq \mu(v) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have shown that

$$\limsup_{n\to+\infty} c(u;n)^{\frac{1}{n}} \leq \limsup_{n\to+\infty} c(v;n)^{\frac{1}{n}}.$$

By symmetry, we see that this is an equality. Hence, we have shown that the connective constant is independent of the choice of $x \in \mathcal{C}$. □

3 Proof of the Main Theorem

Lemma 2 (Lemma 2.1, [CS16]). The quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}(x)$ is almost surely constant in x.

Proof. We will show that

$$\hat{\chi}_{h,\beta,\mathbf{X}}(u) \leq \hat{\chi}_{h,\beta,\mathbf{X}}(v)^2 + e^{h+\beta X_{(v,u)}}\hat{\chi}_{h,\beta,\mathbf{X}}(v)$$

holds for any adjacent vertices $u, v \in \mathcal{C}$. By symmetry, this bound shows that $\hat{\chi}_{h,\beta,\mathbf{X}}(u)$ and $\hat{\chi}_{h,\beta,\mathbf{X}}(v)$ must be simultaneously finite or infinite. Since $\mathcal{C}$ is connected, this implies that the convergence of $\hat{\chi}_{h,\beta,\mathbf{X}}(x)$ is the same for all $x \in \mathcal{C}$ for some fixed h. Therefore, the quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}(x)$ does not vary with x.

To show this inequality, we first write the quenched susceptibility at u as follows:

$$\hat{\chi}_{h,\beta,\mathbf{X}}(u) = \sum_{\omega\in\Omega(u)} \exp\left\{-\sum_{j=1}^{|\omega|}(h + \beta X_{e_j(\omega)})\right\}\left(\mathbf{1}_{\{v\in\omega\}} + \mathbf{1}_{\{v\notin\omega\}}\right).$$

For any SAW $\omega \in \Omega(u)$ which passes through v, we can split it at v into a part $\omega_1 \in \Omega(u,v)$, and another part $\omega_2 \in \Omega(v)$. Hence, the first term can be bounded as follows:

$$\sum_{\omega\in\Omega(u)\,;\,v\in\omega} \exp\left\{-\sum_{j=1}^{|\omega|}(h + \beta X_{e_j(\omega)})\right\}$$
$$\leq \left(\sum_{\omega_1\in\Omega(u,v)} \exp\left\{-\sum_{j=1}^{|\omega|}(h + \beta X_{e_j(\omega)})\right\}\right)\left(\sum_{\omega_2\in\Omega(v)} \exp\left\{-\sum_{j=1}^{|\omega|}(h + \beta X_{e_j(\omega)})\right\}\right),$$

$$= \hat{G}_{h,\beta,\mathbf{X}}(u,v)\hat{\chi}_{h,\beta,\mathbf{X}}(v),$$
$$\leq \left(\sum_{w\in\mathcal{C}} \hat{G}_{h,\beta,\mathbf{X}}(v,w)\right)\hat{\chi}_{h,\beta,\mathbf{X}}(v),$$
$$= \hat{\chi}_{h,\beta,\mathbf{X}}(v)^2.$$

For SAWs $\omega \in \Omega(u)$ which do not pass through v, we can extend them to v by adding the edge (v,u) at the start of the path to get $\omega' = (v,u) \oplus \omega$. Hence, the second term can be bounded as follows:

$$\sum_{\omega\in\Omega(u)\,;\,v\notin\omega} \exp\left\{-\sum_{j=1}^{|\omega|}(h+\beta X_{e_j(\omega)})\right\} \leq \sum_{\omega'\in\Omega(v)} \exp\left\{-\sum_{j=1}^{|\omega'|}(h+\beta X_{e_j(\omega')})\right\} e^{h+\beta X_{(v,u)}}$$
$$= \hat{\chi}_{h,\beta,\mathbf{X}}(v)e^{h+\beta X_{(v,u)}}.$$

Putting the two terms together, we obtain the desired bound. □

Lemma 3 (analogous to Lemma 2.2, [CS16])**.** The quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}$ is non-random.

Proof. Note that

$$\sum_{q\in\mathbb{Q}\cup\{+\infty\}} \mathbf{P}\left[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq q\right] \geq \mathbf{P}[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq +\infty] = 1.$$

Since the event $\left\{\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq q\right\}$ is measurable with respect to the tail σ-algebra of $(X_e)_{e\in\mathbb{E}}$, by Kolmogorov's 0-1 law, its probability $\mathbf{P}\left[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq q\right]$ is either 0 or 1 for every $q \in \mathbb{Q}\cup\{+\infty\}$.

Let $h = \inf\left\{q \in \mathbb{Q}\cup\{+\infty\}\ ;\ \mathbf{P}\left[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq q\right] = 1\right\}$. If $h = +\infty$, then $\mathbf{P}\left[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} = +\infty\right] = 1$ and we are done. Otherwise, we have $h < +\infty$ and $\mathbf{P}\left[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq h\right] = 1$ by right-continuity. Since h is the infimum, for any $\varepsilon > 0$, $\mathbf{P}\left[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq h-\varepsilon\right] < 1$, which means that it must be zero. Therefore, we have

$$\mathbf{P}\left[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} = h\right] = \mathbf{P}\left[\left\{\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq h\right\} \setminus \bigcup_{\varepsilon\in\mathbb{Q}\cap(0,1)} \left\{\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq h-\varepsilon\right\}\right],$$
$$\geq \mathbf{P}\left[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq h\right] - \sum_{\varepsilon\in\mathbb{Q}\cap(0,1)} \mathbf{P}\left[\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}} \leq h-\varepsilon\right],$$
$$= 1.$$

Hence, we have shown that the quenched critical point $\hat{h}^{\mathsf{q}}_{\beta,\mathbf{X}}$ is almost surely equal to some fixed constant h. □

Now, we show the inequality. Let $\beta > 0$ and $\delta > 0$. Then we want to show that for $h = h_0 - \beta\,\mathbf{E}\left[X_e\right] - \beta\delta$, the quenched susceptibility $\hat{\chi}_{h,\beta,\mathbf{X}}(x)$ is almost surely infinite for all $x \in \mathcal{C}$. Since $\hat{\chi}_{h,\beta,\mathbf{X}}(x)$ is almost surely constant in x, it suffices to show this for a fixed $x \in \mathcal{C}$.

Let $\Delta_e = X_e - \mathbf{E}\left[X_e\right]$. We define a (random) set of SAWs:

$$\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n) = \left\{ \omega \in \Omega(x;n)\,;\ \left| \frac{1}{n}\sum_{j=1}^{n} \Delta_{e_j(\omega)} \right| < \delta \right\}.$$

In particular, for every $\omega \in \hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)$, we have the following bound:

$$\delta - \frac{1}{n}\sum_{j=1}^{n} \Delta_{e_j(\omega)} \geq 0.$$

We can now bound the quenched susceptibility $\hat{\chi}_{h,\beta,\mathbf{X}}(x)$ from below:

$$\begin{aligned}
\hat{\chi}_{h,\beta,\mathbf{X}}(x) &= \sum_{\omega \in \Omega(x)} \exp\left\{ -\sum_{j=1}^{|\omega|} (h_0 - \beta\,\mathbf{E}\left[X_e\right] - \beta\delta + \beta X_{e_j(\omega)}) \right\}, \\
&\geq \sum_{n=0}^{+\infty} \sum_{\omega \in \hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)} \exp\left\{ -\sum_{j=1}^{n} (h_0 - \beta\,\mathbf{E}\left[X_e\right] - \beta\delta + \beta X_{e_j(\omega)}) \right\}, \\
&= \sum_{n=0}^{+\infty} \sum_{\omega \in \hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)} \frac{1}{\mu^n} \exp\left\{ \beta \sum_{j=1}^{n} (\delta - \Delta_{e_j(\omega)}) \right\}, \\
&= \sum_{n=0}^{+\infty} \sum_{\omega \in \hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)} \frac{1}{\mu^n} \exp\left\{ \beta n \underbrace{(\delta - \frac{1}{n}\sum_{j=1}^{n} \Delta_{e_j(\omega)})}_{\geq 0} \right\}, \\
&= \sum_{n=0}^{+\infty} \sum_{\omega \in \hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)} \frac{1}{\mu^n}, \\
&\geq \sum_{n=0}^{+\infty} \frac{1}{\mu^n} \left| \hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n) \right|.
\end{aligned}$$

Notice that if there are many good n-step SAWs, say $\left|\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)\right| \geq \frac{1}{2}c(n)$ for infinitely many n, then the susceptibility diverges.

$$\begin{aligned}
&\mathbf{P}\left[\hat{\chi}_{h,\beta,\mathbf{X}}(x) = \infty\right] \\
&\geq \mathbf{P}\left[\hat{\chi}_{h,\beta,\mathbf{X}}(x) = \infty \mid \limsup_{n\to+\infty} \left\{ \left|\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)\right| \geq \frac{1}{2}c(n) \right\} \right] \mathbf{P}\left[\limsup_{n\to+\infty} \left\{ \left|\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)\right| \geq \frac{1}{2}c(n) \right\} \right],
\end{aligned}$$

$$\geq \limsup_{n\to+\infty} \mathbf{P}\left[\left|\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)\right| \geq \frac{1}{2}c(n)\right].$$

Since the convergence of the quenched susceptibility must depend on the conductances of infinitely many edges, the probability $\mathbf{P}\left[\hat{\chi}_{h,\beta,\mathbf{X}} = \infty\right]$ is either 0 or 1. Therefore, it suffices to show that the above limit is positive. To do this, we will rely on the Paley-Zygmund inequality.

Lemma 4. Let $Z \geq 0$ be a non-negative random variable with finite second moment. Then for any $\varepsilon \in (0,1)$,

$$\mathbf{P}\left(Z \geq \varepsilon\, \mathbf{E}[Z]\right) \geq (1-\varepsilon)^2 \frac{\mathbf{E}[Z]^2}{\mathbf{E}[Z^2]}.$$

Proof. We write

$$\mathbf{E}\left[Z\right] = \mathbf{E}\left[Z\mathbf{1}_{\{Z<\varepsilon\,\mathbf{E}[Z]\}}\right] + \mathbf{E}\left[Z\mathbf{1}_{\{Z\geq\varepsilon\,\mathbf{E}[Z]\}}\right].$$

The first term is bounded by

$$\mathbf{E}\left[Z\mathbf{1}_{\{Z<\varepsilon\,\mathbf{E}[Z]\}}\right] \leq \int_{\{Z<\varepsilon\,\mathbf{E}[Z]\}} \varepsilon\,\mathbf{E}\left[Z\right]\mathrm{d}\mathbf{P} \leq \varepsilon\,\mathbf{E}\left[Z\right].$$

We can apply the Cauchy-Schwarz inequality to the second term:

$$\mathbf{E}\left[Z\mathbf{1}_{\{Z\geq\varepsilon\,\mathbf{E}[Z]\}}\right]^2 \leq \mathbf{E}\left[Z^2\right]\mathbf{P}\left[Z \geq \varepsilon\,\mathbf{E}\left[Z\right]\right].$$

Putting the two terms together, we obtain the desired inequality. □

We apply this to the random variable $Z = \left|\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)\right|$. We have

$$\mathbf{E}\left[\left|\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)\right|\right] = \sum_{\omega\in\Omega(x)\,;\,|\omega|=n} \mathbf{P}\left[\left|\frac{1}{n}\sum_{j=1}^{n}\Delta_{e_j(\omega)} < \delta\right|\right] = c(x;n)\mathbf{P}\left[\left|\frac{1}{n}\sum_{j=1}^{n}\Delta_{e_j} < \delta\right|\right].$$

Since $(X_e)_{e\in\mathbb{E}}$ are i.i.d., $(\Delta_e)_{e\in\mathbb{E}}$ are also i.i.d.. We apply the weak law of large numbers to obtain

$$\mathbf{P}\left[\left|\frac{1}{n}\sum_{j=1}^{n}\Delta_{e_j} < \delta\right|\right] = 1 - o(1) \quad \text{as } n \to +\infty.$$

Combining this with the trivial bound $\mathbf{E}\left[\left|\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)\right|^2\right] \leq c(n)^2$, we have

$$\begin{aligned}\mathbf{P}\left[\left|\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)\right| \geq \frac{1}{2}c(n)\right] &\geq \left(1-\frac{1}{2}\right)^2 \frac{c(x;n)^2\mathbf{P}\left[\left|\frac{1}{n}\sum_{j=1}^{n}\Delta_{e_j} < \delta\right|\right]^2}{c(x;n)^2}, \\ &\geq \frac{1}{4}(1-o(1))^2,\end{aligned}$$

$$\limsup_{n\to+\infty} \mathbf{P}\left[\left|\hat{\Omega}^{\text{good}}_{\delta,\mathbf{X}}(x;n)\right| \geq \frac{1}{2}c(n)\right] \geq \frac{1}{4}.$$

This implies that

$$\mathbf{P}\left[\hat{\chi}_{h,\beta,\mathbf{X}} = \infty\right] = 1$$

for $h = h_0 - \beta\,\mathbf{E}\left[X_e\right] - \beta\delta$ for any $\delta > 0$. Therefore, the quenched critical point $\hat{h}^{\mathfrak{q}}_{\beta,\mathbf{X}}(x)$ satisfies

$$h_0 - \beta\,\mathbf{E}\left[X_e\right] \leq \hat{h}^{\mathfrak{q}}_{\beta,\mathbf{X}}(x) \quad \text{almost surely.}$$

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