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Let $m=2$ $\quad Fl(k) = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!} = \binom{n}{k}_q$.

$\# Gr(k, \mathbb{F}_q^n) = \binom{n}{k}_q$

We have been studying $V = \mathbb{F}^n$ or $\mathbb{R}^n$.

Let X be a polyhedral space. Define

$$\binom{X}{k} := \{ k \text{ distinct points of } X, \text{ no order} \},$$
$$= \{ k\text{-subsets of } X \}.$$

Question: $\chi\left(\binom{X}{k}\right) = ?$

We will consider $\binom{X}{k}$ ~~as~~ as a topological space.

Define $X^k = \underbrace{X \times X \times \cdots \times X}_{k \text{ times}}$

$$= \{ (x_1, \dots, x_k) : x_i \in X \}.$$

Need: distinct and disregard order.

Let $X_{ij} := \{ (x_1, \dots, x_k) \in X^k : x_i = x_j \} \subseteq X^k$.
(assume $i<j$).

~~Consider X_{ij}~~ They are subspaces of X^k.

Let $\mathcal{A} := \{ X_{ij} : i<j \}$,

$$M(\mathcal{A}) := X^k \setminus \bigcup_{i<j} X_{ij}$$

Consider the action $Sym(k) \times M \longrightarrow M$

$$(\sigma, (x_1, \dots, x_k)) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(k)}).$$

Then we see that as sets, $\binom{X}{k} \cong M(\mathcal{A})/Sym(k)$.

We ~~will~~ define $\binom{X}{k}$ as this ~~a~~ topological space ~~with the~~

In AG, they take X to be alg variety, then this this called ~~a~~ the symmetric space.

For $a \in \mathbb{R}$, $k \in \mathbb{N}_0$,

$$\binom{a}{k} := \begin{cases} \dfrac{a(a-1)\cdots(a-k+1)}{k!} & k \geq 1 \\ 1 & k = 0. \end{cases}$$

If $X = \Sigma_g$ is a cpt Riemann surface of genus g, then $\chi\left(\binom{X}{k}\right) = \binom{\chi(X)}{k}$.

($\chi(X)$ not necessarily integer

We can calculate $\chi(M(\mathcal{A})) = \chi(X^k) - \chi\left(\bigcup_{i<j} X_{ij}\right)$

$$= \left(\chi(X)\right)^k - \chi\left(\bigcup_{i<j} X_{ij}\right)$$

$\chi(X_{ij}) = \chi(X^{k-1})$. The union is handled by inclusion-exclusion.

Consider the quotient

$$\mathcal{M}(\mathcal{A}) \xrightarrow{\rho} \binom{X}{k}$$
$$(x_1, \dots, x_k) \mapsto \{x_1, \dots, x_k\}$$

Clearly, $\left| \rho^{-1}(\{x_1, \dots, x_k\}) \right| = k!$

$$\chi(\mathcal{M}(A)) = \chi\left(\binom{X}{k}\right) \cdot k!$$

So $\chi\left(\binom{X}{k}\right) = \dfrac{\chi(\mathcal{M}(A))}{k!}$

Last time we learned q-analogues.

$\mathfrak{S}_n(a_1, \dots, a_m) := \{ \sigma \in \mathrm{Sym}(n) : \text{s.t.}$
$\mathrm{Des}(\sigma) \subseteq \{a_1, a_1+a_2, \dots, a_1 + \dots + a_m\} \}$.

(Given a permutation $\sigma = (a_1 \cdots a_n)$,
$\mathrm{Des}(\sigma) = \{ i : a_i > a_{i+1} \}$.)

Then $$\sum_{\sigma \in \mathfrak{S}_n(a_1, \dots, a_m)} q^{\mathrm{inv}(\sigma)} = \begin{bmatrix} n \\ a_1, \dots, a_m \end{bmatrix}_q.$$

$\mathrm{inv}(\sigma) = \#\{ (i,j) \; 1 \le i < j \le n \; a_i > a_j \}$ — # of inversions

From the definition of $\begin{bmatrix} n \\ a_1, \dots, a_m \end{bmatrix}_q$, we cannot easily see that it is a polynomial in q, but now we can.