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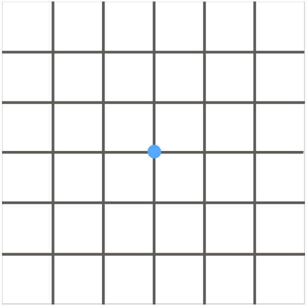

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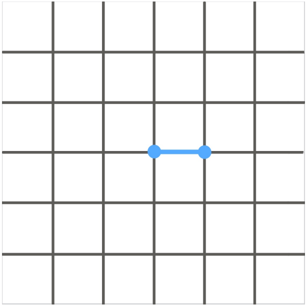

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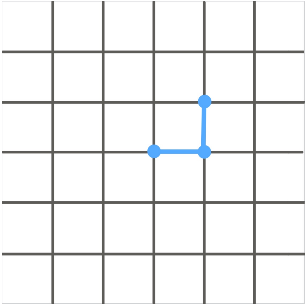

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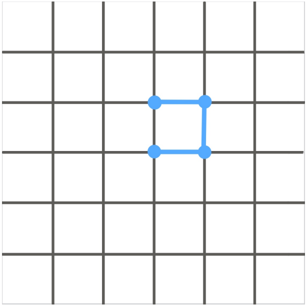

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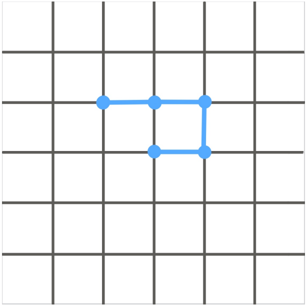

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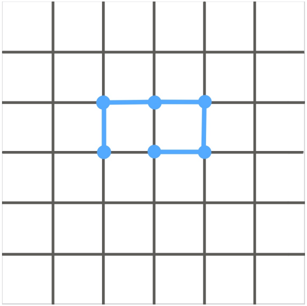

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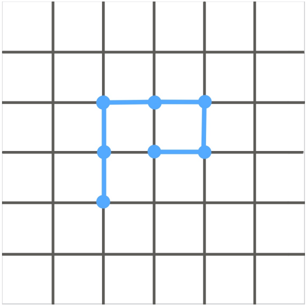

Directed Polymers

The directed polymer is a random walk path of length N "stretched out in time direction". For now, we look at $\mathbb{Z}^d$.

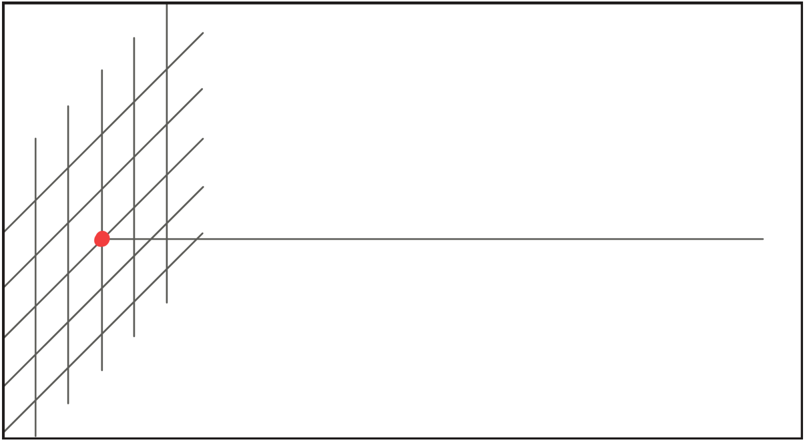

Figure: An instance of directed polymer of time horizon 6, $d = 2$.

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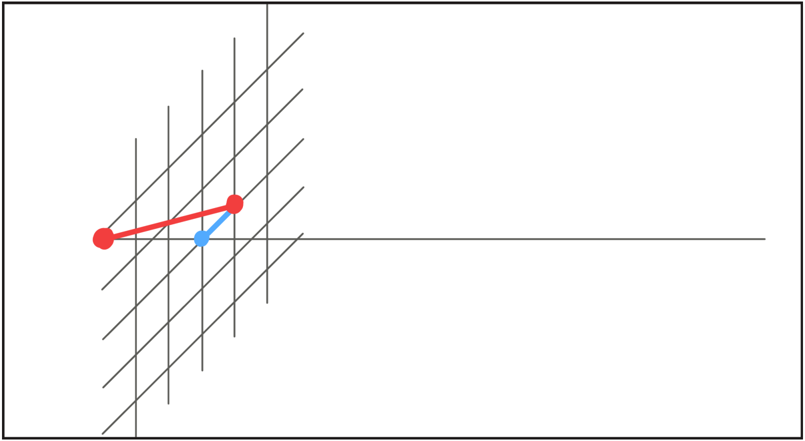

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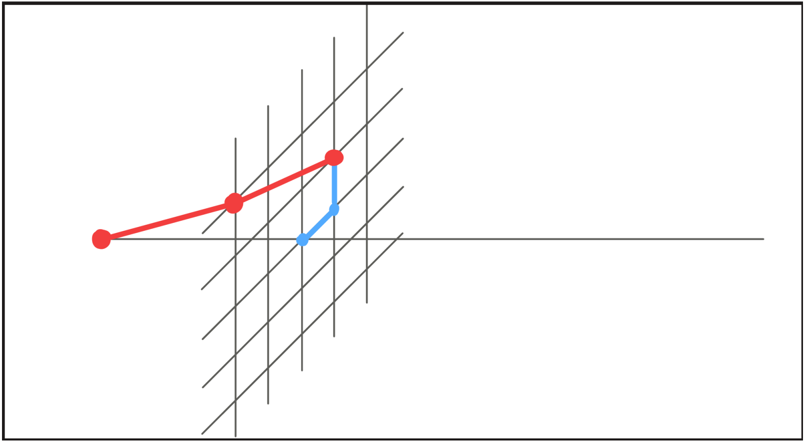

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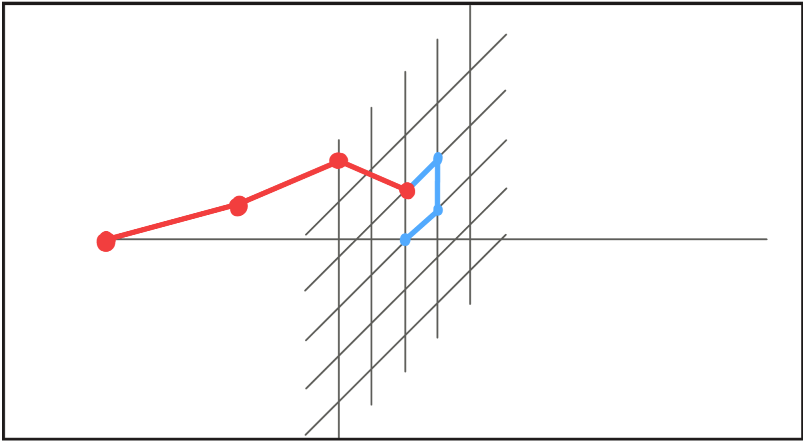

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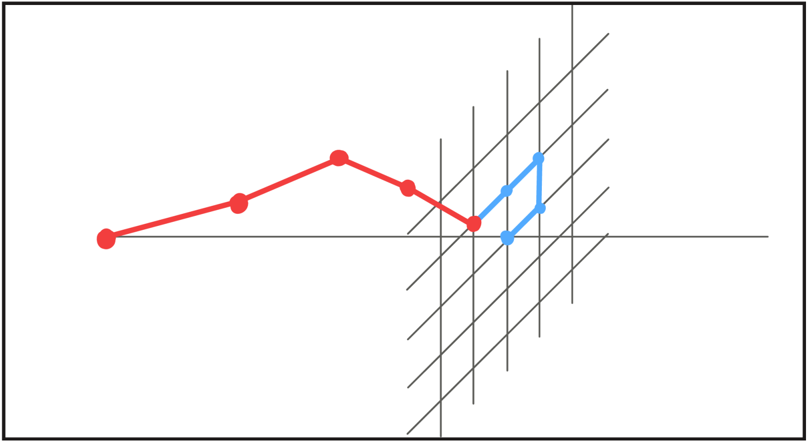

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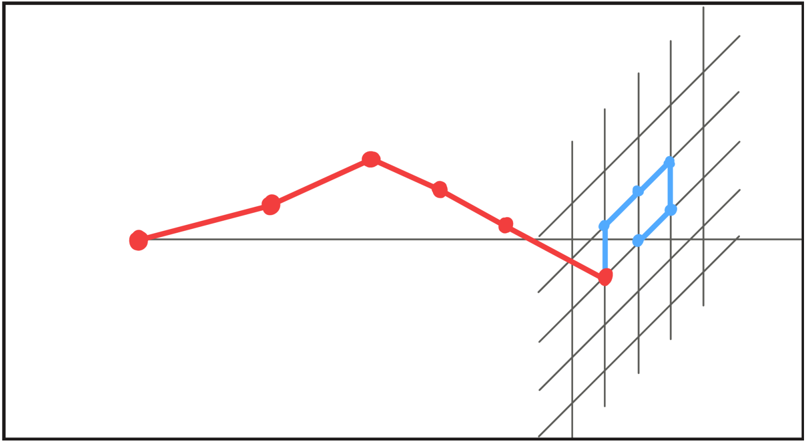

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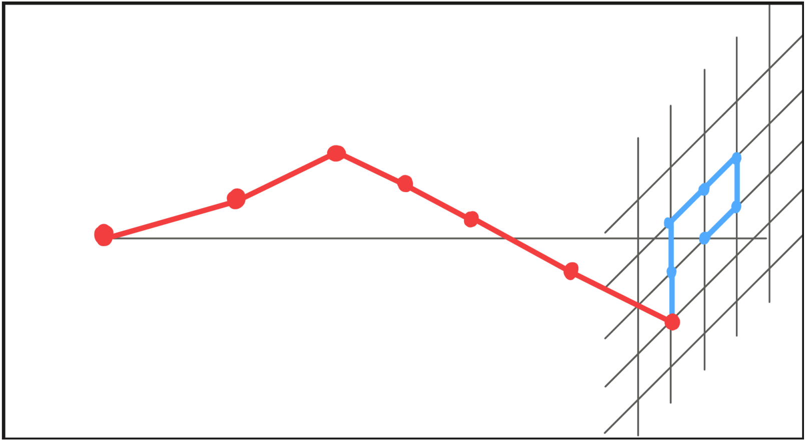

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Formal construction: For a directed polymer of time horizon $N \in \mathbb{N}$, $(X_0, X_1, \dots, X_N)$, we define

1. **Random environment**: A collection of independent, identically distributed random variables with mean 0, variance 1, governed by $\mathbf{P}$

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Random Environment

Formal construction: For a directed polymer of time horizon $N \in \mathbb{N}$, $(X_0, X_1, \ldots, X_N)$, we define

1. **Rewards & penalties**: A collection of independent, identically distributed random variables with mean 0, variance 1, governed by $\mathbf{P}$

$$\left\{\omega(i, x) \,:\, i \in \mathbb{N}_0, x \in \mathbb{Z}^d\right\}$$

2. **Net reward** of a polymer chain: $\sum_{i=0}^{N} \omega(i, X_i)$
3. **Effectiveness**: $\beta \geq 0$
4. **Weight** of a polymer: $\exp\left(\beta \sum_{i=0}^{N} \omega(i, X_i)\right)$
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Directed Polymers in a Random Environment

For a fixed ω, the directed polymer in a random environment (DPRE) measure on $\mathbb{Z}^d$ is given by

$$\mathrm{d}P_{0,\mathbb{Z}^d}^{N,\beta} = \frac{1}{Z_N(\beta,\omega)} \exp\left(\beta \sum_{i=1}^{N} \omega(i, X_i)\right) \mathrm{d}P_{0,\mathbb{Z}^d} \, .$$

- A specific polymer $(X_0, X_1, \dots, X_N)$ is more preferable if its weight $\exp\left(\beta \sum_{i=0}^{N} \omega(i, X_i)\right)$ is larger.
- $Z_N(\beta,\omega)$ for normalisation to probability measure.

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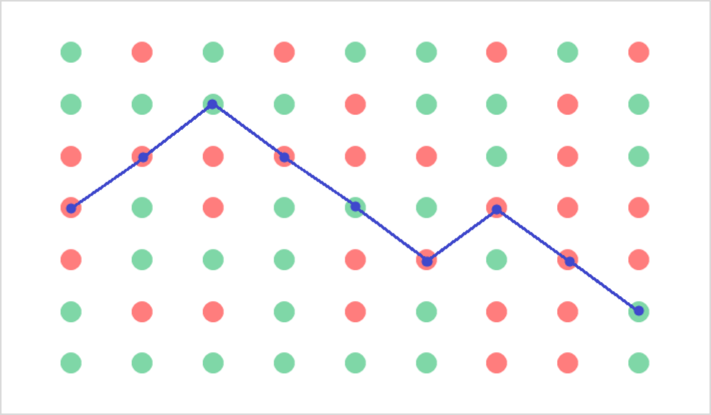
- $Z_N(\beta, \omega)$ is a surprise tool that will help us later.

Example of DPRE

$\omega(i, x)$ at green (red) dot is $+1$ (-1).



Net reward $= -3$
Weight $= \exp(-3\beta)$.

More on Reward/Penalty Mechanism

- Take (for instance) $\omega(i, x) \sim \mathcal{N}(0, 1)$

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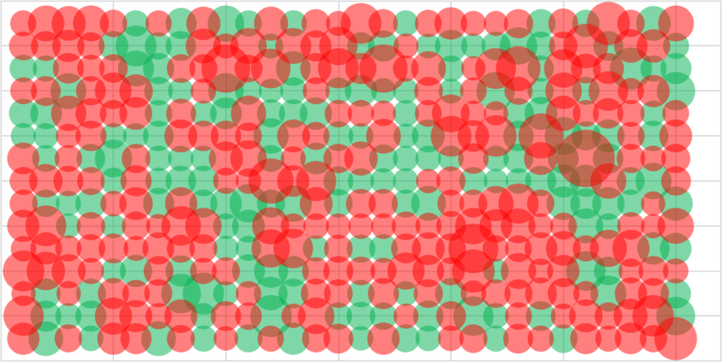
- Harder to be good when β larger.

Random Environments with $d = 1$, $\beta = 0.5$

Generated with $\beta = 0.5$, $\mathcal{N}(0,1)$ random variables. Green (red) bubbles give good (bad) weights. Bubble area $\propto \exp\left(\left|\beta\omega(i,x) - \frac{\beta^2}{2}\right|\right)$.



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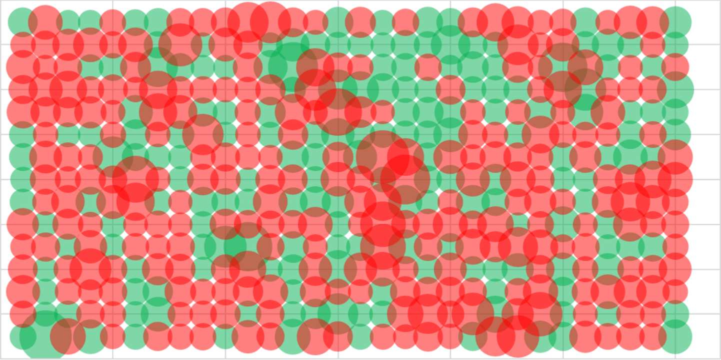

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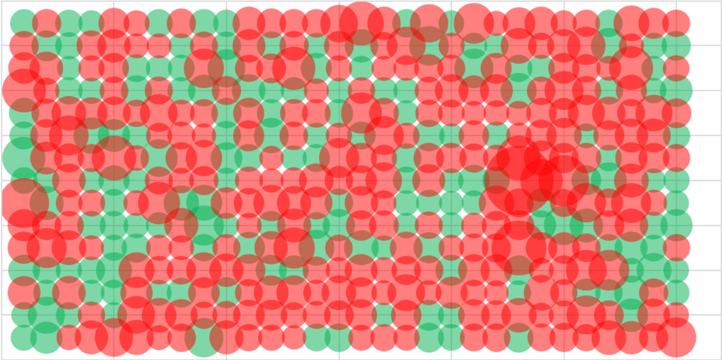

Figure: Example 3, $\beta = 0.5$

Random Environments with $d = 1$, $\beta = 2$

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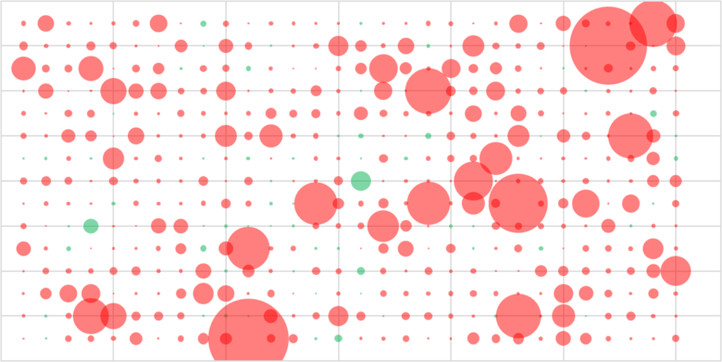

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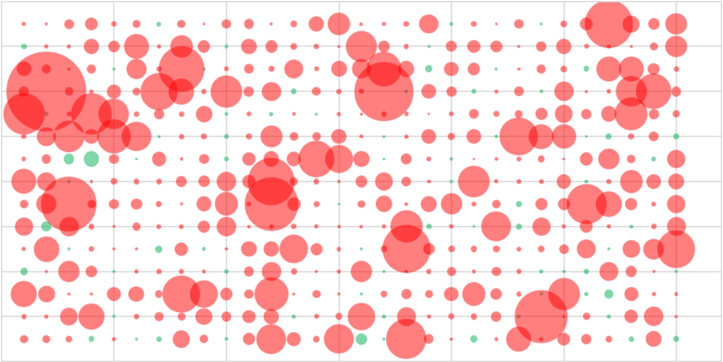

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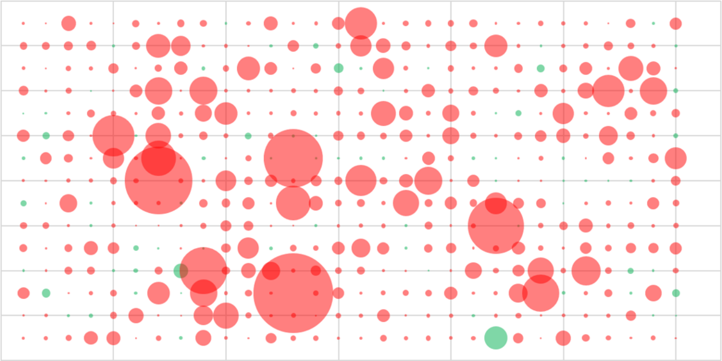

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Localisation of Polymers for larger β

For larger β,

- Take a typical environment ω

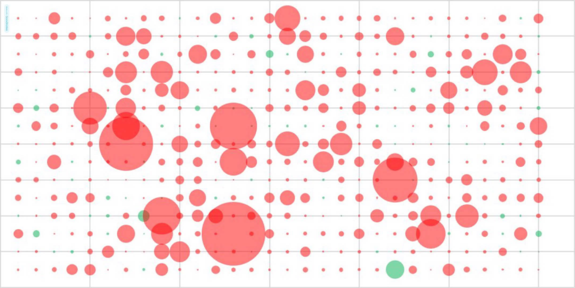

Figure: Typical environment

Localisation of Polymers for larger β

For larger β,

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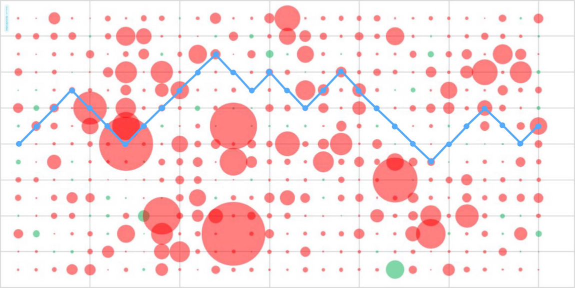

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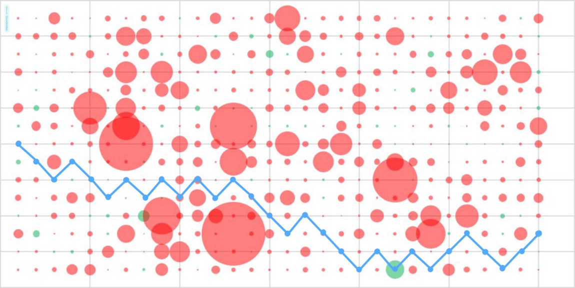

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For larger β,

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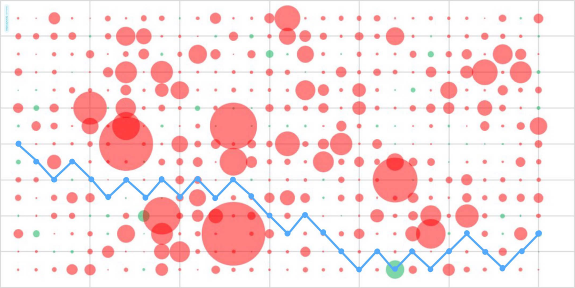
- When sampling with DPRE measure $P_{0,\mathbb{Z}^d}^{N,\beta}$, these polymers are far more likely to be chosen



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Partition Function Viewpoint

$$Z_N(\beta,\omega) = \frac{1}{(2d)^N} \sum_{\text{polymers of length } N} \exp\left(\beta \sum_{i=0}^{N} \omega(i, X_i)\right)$$

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$$\frac{1}{(2d)^N} \sum_{\text{polymers of length } N} \exp\left(\beta \sum_{i=0}^{N} \omega(i, X_i)\right) \ll \mathbf{E}[Z_N(\beta,\omega)]$$

Normalised Partition Function

Define the normalised partition function

$$W_N(\beta, \omega) = \frac{Z_N(\beta, \omega)}{\mathbf{E}\left[Z_N(\beta, \omega)\right]}.$$

Then $\mathbf{E}[W_N(\beta, \omega)] = 1$. It is a martingale.

Normalised Partition Function

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$$W_N(\beta, \omega) = \frac{Z_N(\beta, \omega)}{\mathbf{E}\left[Z_N(\beta, \omega)\right]}.$$

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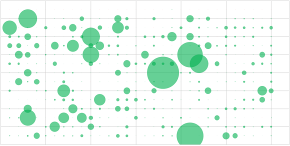
Exceptional environments are the main contributors to $\mathbf{E}[W_N(\beta,\omega)] = 1$.

Exceptional Very Good Environments

For larger β,
Very rarely, we have a very good environment:



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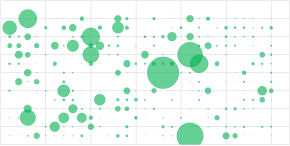

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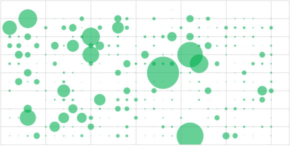

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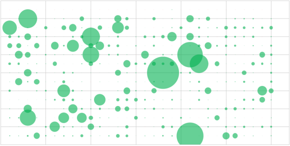

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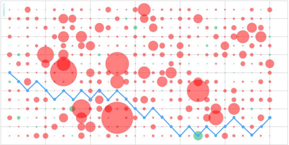
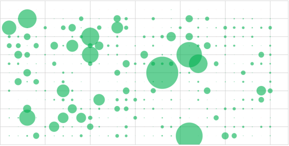
- This ω is main contributor to $\mathbf{E}\left[W_N(\beta,\omega)\right] = 1$.
- These environments become rarer as $N \to \infty$ (need more high rewards).

Recap

For larger β,

- Typical environments: $W_N(\beta, \omega) \ll 1$
    - Dominated by penalties & low rewards (red bubbles)
    - Sprinkled with few high rewards (green bubbles)
    - Localisation to polymers which pass through high rewards
- Exceptional environments: $W_N(\beta, \omega)$ very large
    - Has many high rewards
    - Rarer as $N \to \infty$





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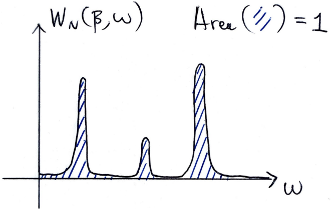
For larger β,
As $N \to \infty$, greater proportion of environments become 'typical'.



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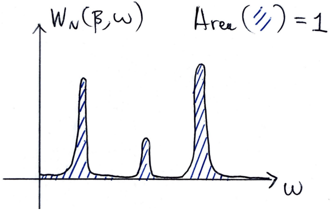

Figure: $W_N(\beta, \omega)$ is concentrated around a few exceptional ωs

For larger β, by the martingale convergence theorem,

$$W_N(\beta, \omega) \xrightarrow{N \to \infty} W_\infty(\beta, \omega) = 0 \quad \mathbf{P}\text{-a.s.}$$

Weak and Strong Disorder

How large do we need β to be to see these effects? Does it suffice for β to be positive?

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Theorem (Imbrie-Spencer'88, Bolthausen'89)

Let $d \geq 3$. Then there exists some $\beta_c \in (0, \infty]$ such that

$$\begin{cases} W_\infty(\beta, \omega) > 0 \quad \mathbf{P}\text{-}a.s. \quad \text{for } \beta < \beta_c, \quad (\text{weak disorder}), \\ W_\infty(\beta, \omega) = 0 \quad \mathbf{P}\text{-}a.s. \quad \text{for } \beta > \beta_c, \quad (\text{strong disorder}). \end{cases}$$

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Theorem (Comets-Shiga-Yoshida'03)

Let $d = 1, 2$. Then for every $\beta > 0$,

$$W_\infty(\beta,\omega) = 0 \quad \mathbf{P}\text{-a.s.}$$