Lecture 1 2026-02-23.

$\vec{u} = \begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix} \in \mathbb{R}^n$ v.s. with std basis $(\vec{e}_i)_1^n$.

$\mathrm{Aut}(\mathbb{R}^n) \equiv GL_n(\mathbb{R})$.

$T \leftrightarrow A = [T\vec{e}_1 \cdots T\vec{e}_n]$

$\mathbb{R}^n = (\mathbb{R}^n$, dot product) std Euclidean v.s.

$\mathbb{R}^{1,n} = (\mathbb{R}^{n+1}$, Lorentz dot product) std Lorentzian v.s.

$u = (u^0, \vec{u})$ $\quad u \cdot v = u^0 v^0 - \vec{u} \cdot \vec{v}$.

$v = (v^0, \vec{v})$

real affine space A of dim n is a principal $\left(\mathbb{R}^n, +\right)$-space, with abelian group

action $A \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ transitive and free

$(p, \vec{u}) \mapsto p + \vec{u}$

If $p + \vec{u} = q$, we write $\vec{u} = q - p$.

E.g. $\mathbb{A}^n_{\mathbb{R}} = \{(x^1, \dots, x^n) \mid x^i \in \mathbb{R}\}$. $(x, \vec{u}) \mapsto x + \vec{u} = (x^1 + u^1, \dots, x^n + u^n)$.

This is the std ~~real~~ <u>real affine space of dim n.</u>

A map $A_1 \xrightarrow{\phi} A_2$ is affine if it preserves affine combination.

affine combination : $\sum c_i p_i$, $\sum c_i = 1$.

sum and scalar multiplication given by point-vector

correspondence: choose O, $p = O + (p - O)$,

~~$(\sum) c^i p_i := O + (\sum) c^i (p_i - O)$ — indep~~

$(\sum c^i p_i) := o + \sum c^i (p_i - o)$ — indep of choice of o.

Einstein notation : $c^i p_i = o + c^i (p_i - o)$.

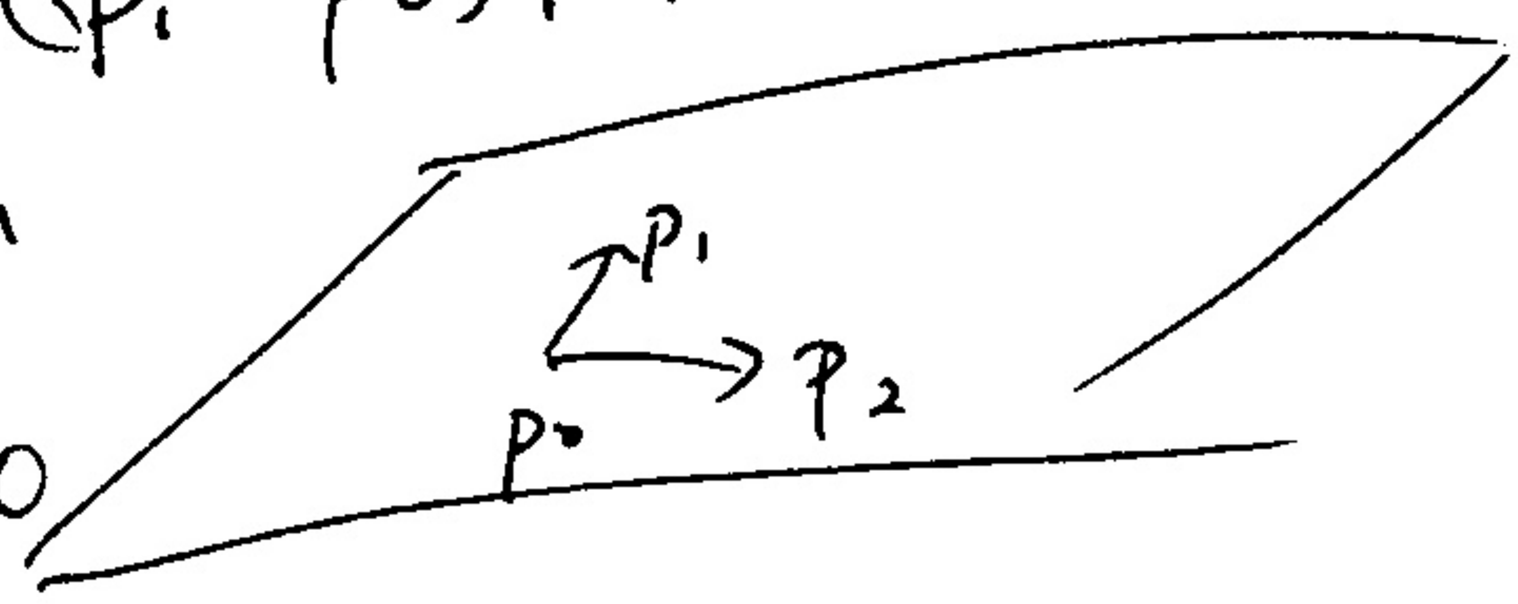
$V_1 \xrightarrow[\cong]{I} \mathbb{R}^n$ — basis

$\mathbb{A}_1 \xrightarrow[\cong]{\phi} \mathbb{A}^n_{\mathbb{R}}$ — frame $(p_0, \dots, p_n)$.

Turns into v.s. with $(p_i - p_0)_1^n$.

$T_o \mathbb{A} = (\mathbb{A}, o) \equiv \mathbb{R}^n$

$p \qquad p - o$



An Euclidean space ~~$\mathbb{R}^n$~~ $\mathbb{E}^n$ of dim n is a homogeneous (principal) $\mathbb{R}^n$-space, i.e. a real affine space with a translation-invariant assignment of inner prod to its tangent space - std Euclidean v.s. $(ds_{\mathbb{E}})^2 = (dx^1)^2 + \dots + (dx^n)^2$.

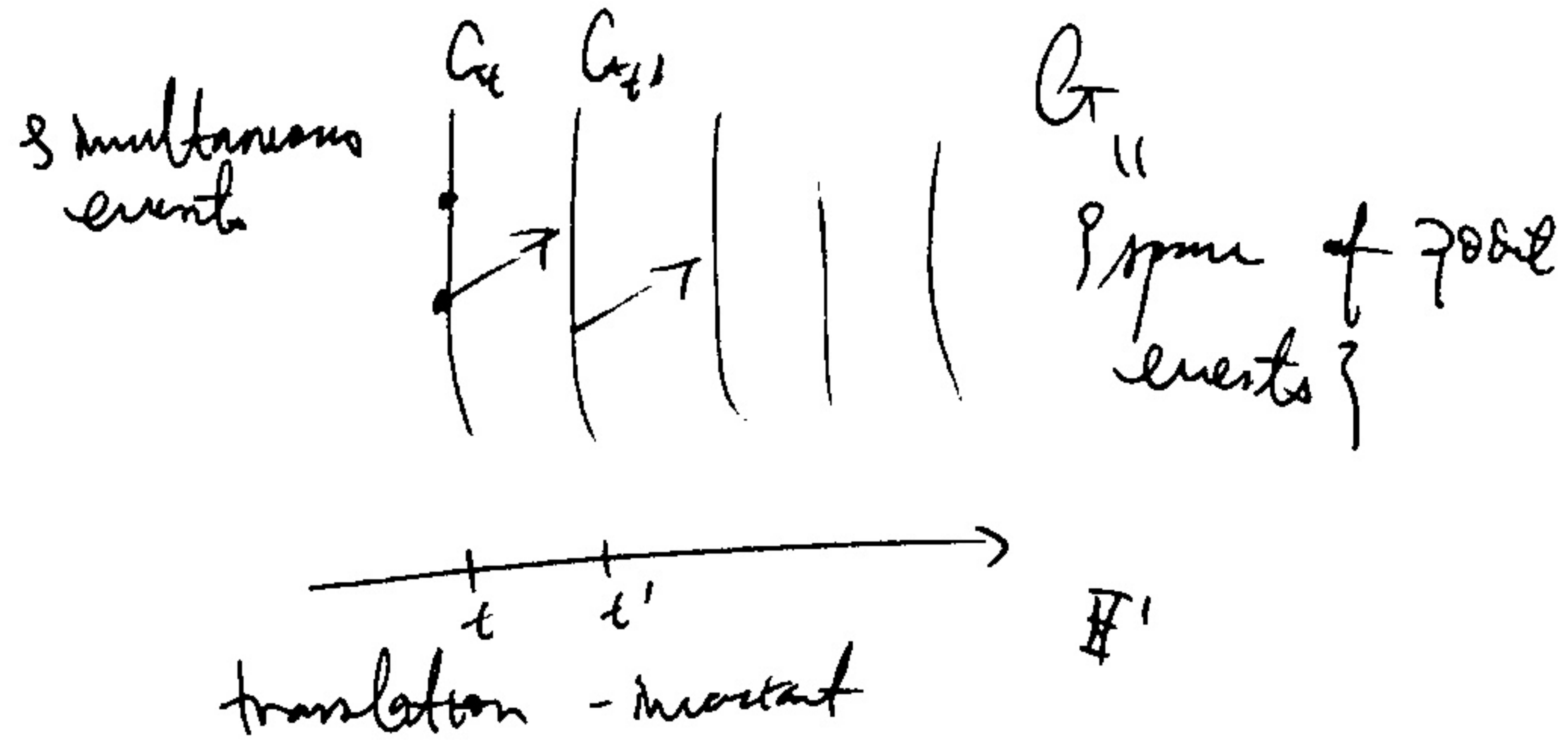
A Lorentz space of dim $1+n$ is a homogeneous $\mathbb{R}^{1,n}$ space, i.e. a real affine space of dim $n+1$ w/ a translation-invariant assignment of Lorentz inner product to its tangent space.

$$(ds_M)^2 = (dx^0)^2 - (dx^1)^2 - \dots - (dx^n)^2$$

A Galilean spacetime G of dim $1+n$ ———— $\mathbb{R}^{n+1}$-space where $\mathbb{R}^{n+1}$ is viewed as a "Galilean v.s." in the sense that

$$0 \to \underbrace{(\mathbb{R}^n, dot)}_{\text{space}} \xrightarrow{\iota} \mathbb{R}^{n+1} \xrightarrow{\pi} \underbrace{(\mathbb{R}^1, dot)}_{\text{time}} \to 0 \quad \text{exact}.$$

$$\begin{array}{ccc} T^{-1}(t) =: G_t & \subseteq & G \\ \downarrow & & \downarrow T \\ t & & \mathbb{E}^1 \end{array}$$



v.s. w/ extra structure	homogeneous space	symmetry group
$\mathbb{R}^n$	affine space	$\mathbb{R}^n \rtimes GL_n(\mathbb{R})$. $(\vec t, A) \leftrightarrow \begin{pmatrix} A & \vec t \\ 0 & 1 \end{pmatrix}$
Euclidean $\mathbb{R}^n$	Euclidean space	$\mathbb{R}^n \rtimes O(n)$
Lorentz v.s.	Lorentzian space	$\mathbb{R}^{n+1} \rtimes O(1,n)$
————	Galileo space	$\mathbb{R}^{n+1} \rtimes Gal(n)$

$$Gal(n) = \mathrm{Aut}\left(0 \to (\mathbb{R}^n, dot) \xrightarrow{\iota} \mathbb{R}^{n+1} \xrightarrow{\pi} (\mathbb{R}^1, dot) \to 0\right)$$

$$\vec x \mapsto \binom{\vec x}{0} \qquad \binom{\vec x}{t} \mapsto t$$

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{R}^n & \xrightarrow{\iota} & \mathbb{R}^{n+1} & \xrightarrow{\pi} & \mathbb{R} & \to & 0 \\ & & \phi_2 \downarrow \text{ preserve Euclidean} & & \phi \downarrow \begin{pmatrix} A & \vec a \\ \alpha & a \end{pmatrix} & & \downarrow \phi_1 \ a \text{ preserve Euclidean} & & \\ 0 & \to & \mathbb{R}^n & \to & \mathbb{R}^{n+1} & \to & \mathbb{R} & \to & 0 \end{array}$$

$$\beta \vec x \mapsto \binom{\beta \vec x}{0} = \binom{A\vec x}{\alpha \vec x}, \quad \binom{A\vec x + \vec v t}{\alpha \vec v + a t} \mapsto \cancel{\alpha \vec x} + at$$

$$\Rightarrow \alpha = 0$$

$$Gal(n) = \left\{ \begin{pmatrix} A & \vec{v} \\ \cancel{0} \, 0 & a \end{pmatrix} : \begin{matrix} A \in O(n) \\ a \in O(1) \\ \vec{v} \in \mathbb{R}^n \end{matrix} \right\}$$

Exercise : $O(1,n) \xrightarrow{c \to +\infty} Gal(n)$.

Manifolds (smooth)

Def An n-manifold M is a subset of a Euclidean space s.t. locally M is smoothly equivalent to $\mathbb{E}^n$. I.e. for any $p \in M$, $(M,p) \cong_{loc} (\mathbb{E}^n, 0)$.

Eg. p (north pole) $\in S^n \subseteq \mathbb{E}^{n+1}$

S^n_{loc} (open nbhd of p in S^n) = upper hemisphere $= \{x^n > 0\} \ni p$.

$\|$

$\mathrm{dom} f = \Gamma_f$, $f = \sqrt{\sum_1^{n-1} (x^i)^2}$

$\|$

$D^n = \mathbb{E}^n_{loc}$

local ~~parametrization~~ : $(M,p) \cong_{loc} (\mathbb{E}^n, 0)$

coordinate system $(M_{loc}, p) \xrightarrow[\cong]{\phi} (\mathbb{E}^n_{loc}, 0) \xrightarrow{Pr^i} \mathbb{R}$

~~x_{p}~~ x^i

M - mnfd

x^i - local coordinates

$\longrightarrow$ local coordinate system

resulting tangent frame

$\frac{\partial}{\partial x^i} (\partial x^i, \partial_i)$

$M_{loc} \xrightarrow[\cong]{(x^1,\dots,x^n)} \mathbb{E}^n_{loc}$

$\longleftarrow$ local parametrization

$\mathbb{E}^n$ ~~canonical~~ std global tangent frame

$(E_1, E_2, \ldots, E_n)$ where $E_i|_p \in T_p\mathbb{E}^n \equiv \{p\}\times\mathbb{R}^n$

$$E_i|_p = (p, \vec{e_i}) \qquad \vec{pq} \equiv (p, q-p)$$

$$M_{loc} \underset{\cong}{\overset{(x^1,\ldots,x^n)}{\rightleftarrows}} \mathbb{E}^n_{loc} \subseteq \mathbb{E}^n$$

$$\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) \longleftrightarrow (E_1, \ldots, E_n)$$

$\Downarrow$

~~(1) resulting tangent frame~~ $\cancel{\frac{\partial}{\partial x^i}}$

resulting cotangent frame dx^i, given by

$$dx^i_p : T_pM \to \mathbb{R} \qquad \text{— dual basis w/ } \left\{\frac{\partial}{\partial x^i}\right\}_1^n.$$

$$\frac{\partial}{\partial x^j}\Big|_p \mapsto \delta^i_j.$$

resulting local coordinate system on $TM : (x^i, \dot{x}^j)$ on TM_{loc}

and resulting local coordinate system on $T^*M = (x^i, p_j)$ on T^*M_{loc}

$$TM = \bigcup_{p\in M} T_pM, \quad T^*M = \bigcup_{p\in M} T^*_pM.$$

$$T_p^*M = (T_pM)^*.$$

Exercise: TM is a manifold.

Not gonna write, see his book Calculus III.