Lecture 5 : Invariance Principle for RCM

<u>Thm</u> (Y_k) RW on RCM, $\mathbb{P}^{\omega}[Y_{k+1}=y \mid Y_k=x] = \dfrac{\omega_{x,y}}{\pi(x)}$.

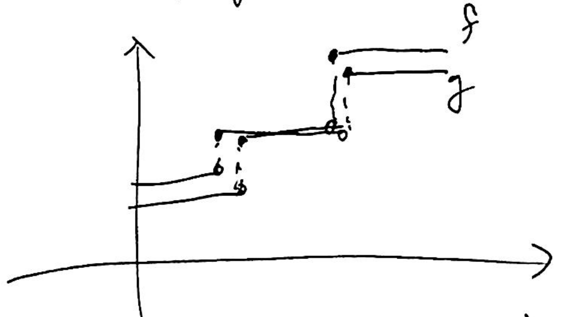
(ω) iid with elliptic condition, then

$$\mathbb{P}\text{-a.s. } Y^{(n)}_t = \begin{cases} \frac{1}{\sqrt{n}} Y_{nt}, & nt \in \mathbb{N}. \\ \text{interpolation} & \text{otherwise} \end{cases}$$

$$(Y^{(n)}_t)_{t\geq 0} \longrightarrow (\sigma B_t)_{t \geq 0} \text{ in } C([0,+\infty)).$$

<u>Rmk</u> Last time, we talked about Markov Jump Processes, ~~which~~ VSRW, CSRW, which are continuous-time. This theorem is ~~true~~ true for those as well, with Skorohod ~~topology~~.

This ~~topology~~ is for ~~cadlag~~ ~~of~~ càdlàg functions.



If we d ~~cont~~ usual norm for these function, then $d(f,g)$ is large.

<u>Prf</u> (Corrector Method).

We don't have a classical CLT, which is why this is difficult. How can we show that the scaling limit is BM?

We want to construct a martingale, which should ~~ex~~ bring ~~us~~ us closer to BM (characterized by continuous martingale with quadratic variation t — Lévy's theorem).

Step 1 (Correctors): $(\phi_j)_{j=1,\dots,d}$ s.t. $\underbrace{e_j \cdot x + \phi_j(x)}_{\Psi_j(x)}$ is harmonic under $\mathcal{L}$. That is,

$$\left(\mathcal{L} f(x) = \sum_{y \sim x} \frac{\omega_{x,y} \left(f(y) - f(x) \right)}{\pi(x)} . \right)$$

$\mathcal{L}(e_j \cdot x + \phi_j(x)) = 0$.

$\Longrightarrow$ $\Psi_j(Y_k)$ is a martingale.

$\cancel{\mathbb{E}[\Psi_{j+1}]}$ $\mathbb{E}\left[\Psi_j(Y_k) - \Psi_j(Y_{k-1}) \mid \mathcal{F}_{k-1} \right]$

$= \mathcal{L} \Psi_j(Y_{k-1}) = 0$.

<u>Step 2</u>: Martingale convergence.

$$\frac{1}{\sqrt{n}} \Psi_j \left(Y_{\lfloor nt \rfloor} \right) \longrightarrow (\sigma B_t)_{t \geq 0}$$ (convergence in entries. Cramer-Wald $\Rightarrow$ Convergence in vector.)

<u>Step 3</u>: $$\sup_{[0,T]} \frac{1}{\sqrt{n}} \left| \phi_j \left(Y_{\lfloor nt \rfloor} \right) \right| \xrightarrow{\mathbb{P}} 0$$

Step 3: sufficient condition: sublinearity of corrector.

$\forall L$, $$\frac{1}{\sqrt{n}} \max_{B_{\sqrt{n} L}} \left| \phi_j(x) \right| \xrightarrow[n \to +\infty]{} 0. \qquad \text{— (1)}$$

$$\left(\text{or} \quad \frac{1}{N} \max_{B_N} \left| \phi_j(x) \right| \xrightarrow[N \to +\infty]{} 0 \right)$$

We are trying to correct a linear function to harmonic, so of course the corrector should not be the dominant term. Otherwise, we are not 'correcting', we are changing/replacing.

If $\phi_j(x)$ linear, then $\max_{B_{\sqrt{n}L}} |\phi_j(x)| \sim \sqrt{n} \notin o(\sqrt{n})$.

~~we don't~~ So we want $\sqrt{n}$ to dominate.

Prf (informal). Thus $Y_{\lfloor nt \rfloor} \sim BM$. So in $[0, \lfloor nt \rfloor)$, $Y_{\lfloor nt \rfloor}$ should be around $\sqrt{\lfloor nt \rfloor}$. So ~~$\sup_{[0,T]} \frac{1}{\sqrt{n}}$~~ the value of $Y_{\lfloor nt \rfloor}$ should be similar to ~~the~~ the value in the ball $B_{\sqrt{n}L}$ for some L. So it should be ~~quite~~ quite natural for us to ~~q~~ try to prove ~~show~~ ~~as this.~~ (1)

Prf We will use a stopping time argument.

Let $T_{n,L} = \inf\{k : Y_k \notin B_{\sqrt{n}L}\}$.

$$\mathbb{P}\left[\sup_{[0,T]} \frac{1}{\sqrt{n}} |\phi_j(Y_{\lfloor nt \rfloor})| > \delta\right]$$

$$= \mathbb{P}[\cdots \delta, T_{n,L} \le T] + \mathbb{P}[\cdots \delta, T_{n,L} > T].$$

The second term is easier. It ~~me~~ is when $Y_{\lfloor nt \rfloor}$ is still in $B_{\sqrt{n}L}$ so the maximum must be ~~&~~ less than that on the ball.

$$\mathbb{P}[\cdots | T_{n,L} > T] \le \mathbb{P}\left[\max_{B_{\sqrt{n}L}} \left|\frac{1}{\sqrt{n}} \phi_j(x)\right| > \delta\right].$$

$$\xrightarrow{n \to \infty} 0$$ ~~...~~ Actually, ~~the~~ this convergence is $\mathbb{P}$-a.s..

The approximation of the first term requires a theorem about martingales.

$$\Psi_j(Y_k) = e_j \cdot Y_k + \phi_j(Y_k)$$

$$\frac{1}{\sqrt{n}}\Psi_j(Y_{\lfloor nt \rfloor}) = \frac{1}{\sqrt{n}} e_j \cdot Y_{\lfloor nt \rfloor} + \frac{1}{\sqrt{n}}\phi_j(Y_{\lfloor nt \rfloor}) . \qquad \text{—— (2)}$$

If $T_{n,L} \leq T$, the our RW is quite ~~fast~~ fast.
It reaches the boundary of $B_{\sqrt{n}L}$ quite quickly.
But as we know, the corrector is small at the boundary.
Because when n is large, at the moment when the RW reaches the boundary, the contribution from the corrector is small
So the martingale part is quite large. So if we estimate the martingale, then we can control in this period the corrector.

$$\mathbb{P}\left[\sup_{[0,T]} \left|\frac{1}{\sqrt{n}}\phi_j(Y_{\lfloor nt \rfloor})\right| < \delta ,\ T_{n,L} \leq T\right]$$

$$\leq \mathbb{P}\left[\cdots\ T_{n,L} \leq T,\ \frac{1}{\sqrt{n}} \max_{B_{\sqrt{n}L}} |\phi_j| \geq 1\right]$$

$$+ \mathbb{P}\left[\cdots\ \cdots\ \cdots\ \cdots \qquad \leq 1\right]$$

The former term also $\to 0$. Why? Because

Still look at the large
Notice $(*)$. We let $\lfloor nt \rfloor = T_{n,L}$. Then what happen? $\frac{1}{\sqrt{n}}\phi_j(Y_{T_{n,L}}) \leq 1$ for large n.

We also know $\frac{1}{\sqrt{n}} e_j \cdot Y_{\lfloor nt \rfloor} \approx 1$.

So we know the size of the martingale.

$$\mathbb{P}\left[\sup_{[0,T]} \left| \frac{1}{\sqrt{n}} \Psi_j (Y_{\lfloor nt \rfloor}) \right| \geq L - 1 \right]$$

$$\leq \frac{1}{L-1} \mathbb{P}[\cdots]$$ Doob maximal inequality

$$\xrightarrow{L \to +\infty} 0 .$$

Basically we truncate to ~~obta~~ add the information about if $Y_{\lfloor nt \rfloor} \in \cancel{B_{\sqrt{n}}} B_{\sqrt{n} L}$.

So we proved a property of the corrector. But we have not talked about its existence ...

Now, we talk about some martingale convergence ~~theorem thing~~.

Step 2: If we have a martingale, we create a prob. space, BM, and stopping time. then BM | stopping time is the same as our martingale.

Thm (Embedding).

Let (M_k) be a L^2 martingale. Then $\exists$ BM $(B_t)_{t \geq 0}$, and stopping times T_k s.t.

$$(M_k)_{k \in \mathbb{N}} \stackrel{d}{=} (B_{T_k})_{k \in \mathbb{N}} .$$

Why is this a embedding? We embed a martingale into a BM.

<u>Idea</u> We do a scaling transformation.

$$\left(M^{(n)}_t\right) = \left(\frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor}\right)_{t\geq 0} \overset{d}{=} \left(\frac{1}{\sqrt{n}} B_{T_{\lfloor nt \rfloor}}\right)_{t\geq 0}.$$

Then in some sense, we want to show that this converges in law to $(B_t)_{t\geq 0}$. Is $T_{\lfloor nt \rfloor}$ ~~to~~ close to nt? This is ~~similar to~~ the scaling invariance of BM.

Try again. We have a martingale. We know that it can be embedded into the path of a BM. Now, however. His lecture is messy.

(This theorem from Durrett.).

<u>e.g</u> Define X, $\mathbb{E}[X]=0$, $X\in\{-a,b\}$. $a,b>0$

$\Rightarrow \mathbb{P}[X=b] = \frac{a}{a+b}$, $\mathbb{P}[X=a] = \frac{b}{a+b}$.

We first construct ~~a~~ X from BM. $\mathbb{E}[\tau_{-a}\wedge\tau_b]$

— Gambler's ruin? $= \mathbb{E}[B^2_{\tau_{-a}\wedge\tau_b}]$

$= \mathbb{E}[X^2] = ab.$

$$B_{\tau_{-a}\wedge\tau_b} \overset{d}{=} X.$$

<u>e.g.</u> General X, $\mathbb{E}[X]=0$, $\mathrm{Var}[X]<+\infty$.

$$C := \int_0^{+\infty} x\, dF_X(x) = -\int_{-\infty}^0 x\, dF_X(x).$$

Given φ a test function, $C\mathbb{E}[\varphi(x)]$ ~~=~~

~~$= \int_{-\infty}^0 \int_0^{+\infty} \varphi(x)\, dF_X(x)\, dF_X(y) \int_0^{+\infty} \int_{-\infty}^0 \varphi(y)\, dF_X(x)\, dF_X(y)$~~

$y \in (-\infty, 0]$
$x \in [0, +\infty)$

$$= \int_{-\infty}^{0} -y \int_{0}^{+\infty} \varphi(x)\, dF_X(x)\, dF_X(y) + \int_{0}^{+\infty} x \int_{-\infty}^{0} \varphi(y)\, dF_X(y)\, dF_X(x)$$

$$= \int_{-\infty}^{0} \int_{0}^{+\infty} (x-y)\left(\frac{x}{x-y}\varphi(y) - \frac{y}{x-y}\varphi(x)\right) \quad dF_X(x)\, dF_X(y)$$

$$\Rightarrow \mathbb{E}[\varphi(X)] = \int_{-\infty}^{0}\int_{0}^{+\infty} \frac{(x-y)}{c} \underbrace{\left[\frac{x}{x-y}\varphi(y) + \frac{y}{x-y}(x)\right]}_{\text{2-point distribution}} dF_X(x)\, dF_X(y).$$

$\frac{x-y}{c}$ ~ density function

$\mathbb{E}[\varphi(B_{\tau_y \wedge \tau_x})]$.

Of course, talking about ~~stopping~~ time requires us to include sampling X and y into filtration. This is not in the natural filtration.

~~Pf~~ For (M_k). First ~~we~~ construct M_1.

Now, conditioning on M_1, construct M_2, etc. ☺.

Pf (Actual, maybe).

Given X r.v. with $\mathbb{E}[X] = 0$, $\mathbb{E}[X^2] < +\infty$.

We know how to construct: $B_T \overset{d}{=} X$.

Suppose we have $(B_{T_1}, \dots, B_{T_{k-1}}) \overset{d}{=} (M_1, \dots, M_{k-1})$.

~~We let~~ Define X r.v.: $\mathbb{P}[X \in \cdot] = \mathbb{P}[\underbrace{M_k - M_{k-1}}_{\text{to centre}} \in \cdot \mid M_1, \dots, M_{k-1}]$.

Then the above lets us sample an r.v.

$$B_{T_{k-1}+T_k} - B_T \overset{d}{=} X \overset{d}{=} M_k - M_{k-1} \text{ conditioned on } M_1, \dots, M_{k-1}.$$

Strong Markov Property: $B_{\underbrace{T_{k-1}+T}_{:=T_k}} - B_T$ is BM. $\blacksquare$

Thm (Martingale CLT).

(M_k) L^2-martingale, $\Delta M_k = M_k - M_{k-1}$

If ① $V_n = \sum_{k=1}^{n} \mathbb{E}[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}]$

(quadratic variation

$\lim_{n\to\infty} \frac{V_n}{n} = \sigma^2$.

similar to tightness — ② $\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}[(\Delta M_k)^2 \mathbb{1}_{|\Delta M_k| \geq \varepsilon\sqrt{n}} \mid \mathcal{F}_{k-1}] \to 0$.

increment cannot be too large.

$$S_t^{(n)} := \begin{cases} \frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor} & nt \in \mathbb{N} \\ \text{interpolation} & \text{otherwise.} \end{cases}$$

$$(S_t^{(n)})_{t\geq 1} \xrightarrow{n\to+\infty} (B_t)_{t\geq 0} \text{ in } C([0,+\infty))$$

We will use the Embedding Theorem.

$\Delta T_k = T_k - T_{k-1}$. $\mathbb{E}[(B_{T_k} - B_{T_{k-1}})^2 \mid \mathcal{F}_{k-1}]$

$= \mathbb{E}[T_k - T_{k-1} \mid \mathcal{F}_{T_{k-1}}]$

$= \mathbb{E}[(M_k - M_{k-1})^2 \mid \mathcal{F}_{k-1}]$

$= \mathbb{E}[(\Delta M_k)^2 \mid \mathcal{F}_{k-1}]$.

$\frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor}$

$\downarrow$

$\frac{1}{\sqrt{n}} B_{T_{\lfloor nt \rfloor}}$

$\overset{d}{=}$

B_t

The increase of stopping times is approx. linear

I gave up.

Step I: Construction of corrector.

There will be a lot of details which will be left to be gradually filled in in later lectures.

Today, we will talk about the construction and ~~and~~ sublinearity will be left for later.

There are a few methods / styles for the construction, each with their own pros and cons.

Def $(\phi_j)_{j=1,\dots,d}$ s.t. $\mathcal{L}_V(e_j \cdot x + \phi_j(x)) = 0 \ \forall x \in \mathbb{Z}^d$.

We don't really care about its uniqueness (if any) since we just want a martingale.

¶ One difficulty is the $(\forall x \in \mathbb{Z}^d)$ part.

* If we only require this for $x \in B_r \cap \mathbb{Z}^d$. This is easy. We can just resolve the discrete harmonic equation in B_r. $\rightsquigarrow \phi_j^{(r)}(x)$.
* Why is unbounded domain problematic? We want to look at this equation in a function space. If we put this in unbounded spaces, then the linear function is ~~easi~~ not in the space $H(\mathbb{R}^d)$.

Is it possible that $\phi_j^{(r)} \xrightarrow{r \to +\infty} \phi_j$ in some sense? This is a good idea which will appear throughout later parts of this course. This is quite a PDE approach.

Another method is just to find the function. But its weird sort of idk what he is saying.

Construction by random field

"derivative about environment"

$$\nabla_x \varphi(\omega) := \varphi(T_x \omega) - \varphi(\omega) \quad , \varphi : \Omega \to \mathbb{R}^d$$

We ~~drag~~ drag the random environment a bit and take the difference.

potential field. $\underline{L^2_{p.t}}$, with finite $\langle\cdot,\cdot\rangle$

$$\langle \varphi, \varphi \rangle = \mathbb{E}\left[\sum_x \omega_{0,x} |\nabla_x \varphi(\omega)|^2 \right]$$

similar ~~th~~ to Dirichlet energy.

then $\exists$ "!" * with T&C. minimizer for the problem

$$\varphi \mapsto \mathbb{E}\left[\sum_x \omega_{0,x} | x + \nabla_x \varphi(\omega)|^2 \right] \quad \text{---} (*)$$

over shift ~~invariant~~ covariant field.

$$F : \Omega \times \mathbb{Z}^d \longrightarrow \mathbb{R}^d.$$

Shift covariance: $F(\omega, x+z) - F(\omega, x)$

$$= F(T_x \omega, z)$$

$x + z - z = x$

$\nabla_{x+z} \varphi(\omega) - \nabla_x \varphi(\omega)$

$$= \varphi(T_{x+z}\omega) - \varphi(T_x \omega)$$

$$= \nabla_z \varphi(T_x \omega).$$

Ok, we do this optimisation problem. ~~then the minimiser~~

with

Pf The functional is positive. So if $\varphi = 0$, then already has some upper bound on the value. So existence ✓.

"Uniqueness": ~~&~~ Make it unique with equivalence relation.

Let ~~ϕ~~ ϕ be minimiser of $(*)$. then for any $\varphi : \Omega \to \mathbb{R}^d$, locally L^2, $\frac{d}{d\varepsilon} \mathbb{E}\left[\sum_x w_{0,x} \left| x + \nabla_x \phi(\omega) + \varepsilon \nabla_x \varphi(\omega)\right|^2\right] = 0$.

$\Rightarrow$ ~~$\langle \phi, \varphi \rangle = 0$~~.

$$\mathbb{E}\left[\sum_x w_{0,x} \underbrace{(\nabla_x \varphi(\omega))}_{= \varphi(T_x\omega) - \varphi(\omega)} (x + \nabla_x \phi(\omega))\right] = 0.$$

$$\mathbb{E}\left[\sum_x w_{0,x}\, \varphi(T_x\omega)\, (x + \nabla_x \phi(\omega))\right] - \mathbb{E}\left[\varphi(\omega) \underbrace{\sum_x x + \nabla_x \phi(\omega)}_{\text{"="}\, L_\omega (x+\phi)}\right] = 0$$

$$\nabla_x \phi(\omega) = T_x(\phi(T_x\omega) - \phi(\omega))$$

$$= \mathbb{E}\left[\sum_x w_{0,x}\, \varphi(x) \left(\overset{-(-x)}{\cancel{x}} + \underbrace{\phi(\omega) - \phi(T_{-x}(\omega))}_{-\nabla_{-x}\phi(\omega)}\right)\right]$$

$$= -2\,\mathbb{E}[\varphi L_\omega(x+\phi)]$$

L_ω is a little different to L_υ. Since φ is any function, you can do a shift on the environment.

$T_y L_\omega(x+\phi) = 0$.

$\overset{?}{\Longrightarrow}$ ~~$T_y L_\omega(x+\phi) = 0$~~ $\mathcal{F}$ $L_\upsilon(e_j \cdot x + \phi_j(x)) = 0$.

We ~~will~~ need to transfer this derivative about the environment ~~to~~ into the derivative about the space. We will not expand on this.