

Week 3 - Random Walks in Random Environments

Baining Zhang

October 5, 2026

These notes are based on Lecture 1 from Part One of *Ten Lectures on Random Media* by Alain-Sol Sznitman. A presentation following an earlier draft was given as part of *MATH 6907Z - Reading Course: Random Walks in Random Environment* at The Hong Kong University of Science and Technology. These notes were then typeset and corrected with significant use of AI (GPT-6 Astra).

0 Models under consideration

We review some models introduced in Sznitman's introduction and set up notation.

- **Site Randomness:** Fix $\kappa \in (0, \frac{1}{2d})$ and let $S = \{u \in \mathbb{Z}^d : |u| = 1\}$.

$$\Omega = \mathcal{P}_{\kappa}^{\mathbb{Z}^d}, \quad \mathcal{P}_{\kappa} = \left\{ (p(u))_{u \in S} \text{ with } p(u) \in [\kappa, 1], \text{ and } \sum_{u \in S} p(u) = 1 \right\}.$$

- **Bond randomness:** Fix $0 < a < b$,

$$\Omega = [a, b]^{\mathbb{E}^d}, \quad \text{with } \mathbb{E}^d \text{ edges of } \mathbb{Z}^d.$$

We use ω to denote a member of Ω , i.e. an environment. We write

$$\begin{aligned} \text{for site: } \omega(y, u) &= [\pi_y(\omega)]_u, & y \in \mathbb{Z}^d, u \in S, \\ \text{bond: } \omega(\{y, y+u\}) &= \pi_{\{y, y+u\}}\omega, & y \in \mathbb{Z}^d, u \in S. \end{aligned}$$

We endow these spaces with the product σ -algebra $\mathcal{F}$, and a product measure $\mathbb{P}$, such that $(\pi_e \omega)_{e \in \mathbb{E}^d}$ (bond) and $(\pi_x \omega)_{x \in \mathbb{Z}^d}$ (site) are iid random variables.

Remark 1. $(\omega(y, u))_{u \in S}$ cannot be independent (unless they are deterministic), since within each vertex, $\omega(x, u)$ are bound by $\sum_{u \in S} \omega(x, u) = 1$.

We write t_x for the canonical translations on Ω , given by

$$\begin{aligned} (t_x \omega)(y) &= \omega(x+y) & (\text{site}), \\ (t_x \omega)(\{y, y+u\}) &= \omega(\{x+y, x+y+u\}) & (\text{bond}). \end{aligned}$$

t_x preserves $\mathbb{P}$, or $\mathbb{P}$ is t_x -invariant, in the sense that

$$\mathbb{P}[A] = \mathbb{P}[t_x^{-1} A].$$

This is by iid-ness.

Proposition 1. $(\Omega, \mathcal{F}, \mathbb{P}, (t_x)_{x \in \mathbb{Z}^d})$ is ergodic, i.e. for every translation invariant (i.e. t_x -invariant for every $x \in \mathbb{Z}^d$) event $A \in \mathcal{F}$,

$$\mathbb{P}[A] \in \{0, 1\}.$$

We show this by showing a stronger property, that this system is (strongly) mixing.

Proposition 2. For any two events $A, B \in \mathcal{F}$,

$$\lim_{|x| \rightarrow +\infty} \mathbb{P}[A \cap t_x B] = \mathbb{P}[A]\mathbb{P}[B].$$

We first show ergodicity from mixing.

Proof. For every translation invariant A ,

$$\mathbb{P}[A] = \mathbb{P}[A \cap t_x A], \quad \mathbb{P}[A] = \lim_{|x| \rightarrow +\infty} \mathbb{P}[A \cap t_x A] = (\mathbb{P}[A])^2 \Rightarrow \mathbb{P}[A] \in \{0, 1\}.$$

□

Proof of Proposition 2. Since $\mathcal{F}$ is generated by (finite-dimensional) projections, for any $\varepsilon > 0$, there exists $A_\varepsilon, B_\varepsilon$, which are supported on finitely many sites/bonds, such that

$$\mathbb{P}[A \Delta A_\varepsilon], \mathbb{P}[B \Delta B_\varepsilon] < \varepsilon.$$

Then $|\mathbb{P}[A \cap t_x B] - \mathbb{P}[A_\varepsilon \cap t_x B_\varepsilon]| \leq 2\varepsilon$. For sufficiently large x ,

$$\begin{aligned} \mathbb{P}[A_\varepsilon \cap t_x B_\varepsilon] &= \mathbb{P}[A_\varepsilon]\mathbb{P}[t_x B_\varepsilon] && \text{(disjoint support)} \\ &= \mathbb{P}[A_\varepsilon]\mathbb{P}[B_\varepsilon] && \text{(translation invariance)}. \end{aligned}$$

Hence, we can write

$$\limsup_{|x| \rightarrow +\infty} |\mathbb{P}[A \cap t_x B] - \mathbb{P}[A_\varepsilon]\mathbb{P}[B_\varepsilon]| \leq 2\varepsilon.$$

We also have

$$|\mathbb{P}[A]\mathbb{P}[B] - \mathbb{P}[A_\varepsilon]\mathbb{P}[B_\varepsilon]| \leq |\mathbb{P}[A](\mathbb{P}[B] - \mathbb{P}[B_\varepsilon])| + |\mathbb{P}[B_\varepsilon](\mathbb{P}[A] - \mathbb{P}[A_\varepsilon])| \leq 2\varepsilon.$$

Combining the above, we have

$$\limsup_{|x| \rightarrow +\infty} |\mathbb{P}[A \cap t_x B] - \mathbb{P}[A]\mathbb{P}[B]| \leq 4\varepsilon.$$

But $\varepsilon > 0$ was arbitrary. So

$$\lim_{|x| \rightarrow +\infty} \mathbb{P}[A \cap t_x B] = \mathbb{P}[A]\mathbb{P}[B].$$

□

We provide some examples of environment measures.

Example 1. Site: At each site, we can

- Sample $\omega(y, u)$ for all $u \in S$ but a fixed $u_* \in S$ according to $\text{Unif}([\kappa, \frac{1}{2d}])$, and set

$$\omega(y, u_*) = 1 - \sum_{\substack{u \neq u_* \\ u \in S}} \omega(y, u).$$

We can make this isotropic by first sampling $u_* \sim \text{Unif}(S)$, then sampling the independent $\text{Unif}([\kappa, \frac{1}{2d}])$ random variables for the other $2d - 1$ directions.

- Take a normalised volume measure on $\mathcal{P}_\kappa$.

Bond: Take $\text{Unif}([a, b])$ on each edge $e \in \mathbb{E}^d$.

We have defined the environments. Let's now define the random walk / Markov chain on a fixed environment ω .

Write $P_{x, \omega}$, for the law of the Markov chain X on $\mathbb{Z}^d$, starting at $x \in \mathbb{Z}^d$, according to the following transition probabilities determined by ω :

$$p^\omega(y, y + u) = \begin{cases} \omega(y, u) & (\text{site}), \\ \frac{\omega(\{y, y + u\})}{\sum_{v \in S} \omega(\{y, y + v\})} & (\text{bond}). \end{cases}$$

Note the following equality for transition probabilities:

$$p^{t_x \omega}(0, u) = p^\omega(x, x + u).$$

1 Environment viewed from the particle

We are ready to define the environment viewed from the particle (EVP). The EVP with initial environment $\omega \in \Omega$ is an Ω -valued process on the probability space for X :

$$\bar{\omega}_n \stackrel{\text{def}}{=} t_{X_n} \omega.$$

Note that $\bar{\omega}$ is governed by $P_{0, \omega}$.

What are the transition probabilities for $\bar{\omega}_n$? For every $\eta \in \Omega$ such that $P_{x, \omega}[\bar{\omega}_n = \eta] > 0$, we have

$$P_{x, \omega}[\bar{\omega}_{n+1} = \zeta \mid \bar{\omega}_n = \eta] = \begin{cases} 0 & \zeta \neq t_u \eta \text{ for any } u \in S, \\ \sum_{v \in S: \zeta = t_v \eta} p^\eta(0, v) & \zeta = t_u \eta \text{ for some } u \in S. \end{cases}$$

We prove something akin to a Markov property under $P_{0, \omega}$ (Sznitman's Proposition 1.1).

Proposition 3. Under $P_{0, \omega}$, for a fixed $\omega \in \Omega$, $\bar{\omega}$ satisfies the Markov property

$$E_{0, \omega}[f(\bar{\omega}_{n+1}) \mid \mathcal{F}_n^{\bar{\omega}}] = E_{0, \bar{\omega}_n}[f(\bar{\omega}_1)] = (Rf)(\bar{\omega}_n), \quad (\text{under } P_{0, \omega}\text{-a.s.})$$

where the transition kernel R is given by

$$(Rf)(\omega) = \sum_{u \in S} p^\omega(0, u)(f \circ t_u)(\omega), \quad \text{for } f \text{ bounded, measurable on } \Omega.$$

Remark 2. One may be more familiar with transition kernel in form $p(\omega, A)$. This can be recovered from R by

$$p(\omega, A) = (R1_A)(\omega) = \sum_{u \in S} p^\omega(0, u) 1_A(t_u \omega).$$

Remark 3. We are conditioning on $\mathcal{F}_n^\omega$, which is, in edge cases, a strict subset of $\mathcal{F}_n^X$. This depends on the initial ω . If ω has any t_u -symmetry, we cannot recover full information on X from $\bar{\omega}$. This does happen with zero $\mathbb{P}$ -probability (unless, say, ω is non-random), though.

Proof. Expanding the expectation yields $E_{0, \bar{\omega}_n}[f(\bar{\omega}_1)] = (Rf)(\bar{\omega}_n)$.

We show the first equality from definition of conditional expectation. Let $A \in \mathcal{F}_n^\omega \subseteq \mathcal{F}_n^X$. We can write

$$1_A = \varphi(X_0, \dots, X_n), \quad \text{for some measurable } \varphi : (\mathbb{Z}^d)^{n+1} \rightarrow \mathbb{R}.$$

So

$$\begin{aligned} E_{0, \omega}[f(\bar{\omega}_{n+1})1_A] &= E_{0, \omega}[f(t_{X_{n+1}}\omega)\varphi(X_0, \dots, X_n)] \\ &= E_{0, \omega}[E_{0, \omega}[f(t_{X_{n+1}}\omega) \mid \mathcal{F}_n^X]\varphi(X_0, \dots, X_n)] \\ &= E_{0, \omega}[E_{X_n, \omega}[f(t_{X_1}\omega)]\varphi(X_0, \dots, X_n)] \quad (\text{Markov property of } X) \\ &= E_{0, \omega}[(Rf)(\bar{\omega}_n)1_A]. \end{aligned}$$

□

We now define the canonical space for the EVP.

Let $\tilde{\Omega} = \Omega^{\mathbb{N}_0}$ denote the canonical space of Ω -valued trajectories. $\tilde{\mathcal{F}}$ the canonical product σ -algebra. $\tilde{\omega}$ the canonical process, and $\tilde{\theta}$ the canonical shift, given by

$$(\tilde{\theta}\tilde{\omega})_n = \tilde{\omega}_{n+1}.$$

Let $\tilde{P}_\omega$ denote the law of the canonical process $\tilde{\omega}$ starting at ω , given by $\tilde{P}_\omega \stackrel{\text{def}}{=} (\bar{\omega})_*(P_{0, \omega})$, with canonical filtration $(\tilde{\mathcal{F}}_n)$.

On the canonical space, we formulate and show the Markov property.

Proposition 4.

$$\tilde{E}_\omega[f(\tilde{\omega}_{n+1}) \mid \tilde{\mathcal{F}}_n] = (Rf)(\tilde{\omega}_n) \quad \tilde{P}_\omega\text{-a.s.}$$

Proof. Let $A \in \tilde{\mathcal{F}}_n$, then there is some measurable $\varphi : \Omega^{n+1} \rightarrow \mathbb{R}$ such that $1_A = \varphi(\tilde{\omega}_0, \dots, \tilde{\omega}_n)$.

$$\begin{aligned} \tilde{E}_\omega[f(\tilde{\omega}_{n+1})1_A] &= \tilde{E}_\omega[f(\tilde{\omega}_{n+1})\varphi(\tilde{\omega}_0, \dots, \tilde{\omega}_n)] \\ &= E_{0, \omega}[f(\bar{\omega}_{n+1})\varphi(\bar{\omega}_0, \dots, \bar{\omega}_n)] \\ &= E_{0, \omega}[E_{0, \omega}[f(\bar{\omega}_{n+1}) \mid \mathcal{F}_n^\omega]\varphi(\bar{\omega}_0, \dots, \bar{\omega}_n)] \\ &= E_{0, \omega}[(Rf)(\bar{\omega}_n)\varphi(\bar{\omega}_0, \dots, \bar{\omega}_n)] \end{aligned}$$

$$= \tilde{E}_\omega[(Rf)(\tilde{\omega}_n)\mathbf{1}_A]. \quad \square$$

Now, we can show the Markov property with any initial law μ :

$$\tilde{E}_\mu[f(\tilde{\omega}_{n+1}) \mid \tilde{\mathcal{F}}_n] = (Rf)(\tilde{\omega}_n), \quad \tilde{P}_\mu\text{-a.s.},$$

where the annealed measure is defined by

$$\tilde{P}_\mu[A] \stackrel{\text{def}}{=} \int \tilde{P}_\omega[A] \mu(d\omega).$$

Then for every $A \in \tilde{\mathcal{F}}_n$,

$$\begin{aligned} \tilde{E}_\mu[f(\tilde{\omega}_{n+1})\mathbf{1}_A] &= \int \tilde{E}_\omega[f(\tilde{\omega}_{n+1})\mathbf{1}_A] \mu(d\omega), \\ &= \int \tilde{E}_\omega[(Rf)(\tilde{\omega}_n)\mathbf{1}_A] \mu(d\omega), \\ &= \tilde{E}_\mu[(Rf)(\tilde{\omega}_n)\mathbf{1}_A]. \end{aligned}$$

One should also show the measurability of $\omega \mapsto \tilde{P}_\omega[A]$. $\square$

Similarly, we can finally finish showing Sznitman's Proposition 1.1 by noting that the annealed Markov property for $\bar{\omega}$ can be shown in a similar way. However, we need to be more careful in defining the annealed measure, as we are trying to specify an initial environment for the EVP on the space for the random walk on this initial environment. We define the annealed measure $P_{0,\mu}$ on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}_0}$

$$P_{0,\mu}[A] = \int P_{0,\omega}[\{X : (\omega, X) \in A\}] \mu(d\omega).$$

We end this part by remarking that, in general, an annealed Markov property does not hold for X . Indeed, the past trajectory provides more information about where the walk will go next, than just knowing the present location. This is due to the past trajectory revealing information about the underlying environment. However, for the EVP, the present step already provides all the information about the environment.

Sznitman remarks that the state space for the EVP, Ω , may be too large for the chain to be useful. His worry, I believe, is in reference to the fact that any non-Markovian process can be made a Markov process if one takes the state space to be the trajectories, and this feels a bit similar. As we will hopefully see, the EVP is useful.

2 Invariant measures

We are interested in the long-time behaviour of X , hence that of the EVP. So we are interested in any invariant measures that it may have. The following definitions are for finite measures only, although in general, invariant measures may be infinite.

Definition 1. Given a Markov chain with kernel R , $\mathbb{Q}$ is invariant for R if

$$\int R h \, d\mathbb{Q} = \int h \, d\mathbb{Q} \quad \text{for every bounded measurable } h.$$

This can be written in the more suggestive form

$$\mathbb{Q}R = \mathbb{Q},$$

where the equality is in terms of linear functionals on the space of bounded measurable functions on Ω , with $\mathbb{Q}f \stackrel{\text{def}}{=} \int f d\mathbb{Q}$.

Remark 4. Note that invariant measures are not unique, since if $\mathbb{Q}$ is an invariant measure for R , then $c\mathbb{Q}$ is also one, for any $c \geq 0$. Moreover, it need not be a probability measure, but if a nonzero finite invariant measure does exist, then one can normalise it to a probability measure.

Some chains admitting invariant measures are said to be reversible.

Definition 2. A Markov chain with transition kernel R , admitting a (nonzero) invariant measure $\mathbb{Q}$, is reversible with respect to $\mathbb{Q}$ if

$$\int h(Rg) d\mathbb{Q} = \int (Rh)g d\mathbb{Q}$$

for every bounded measurable g, h .

Again, a more suggestive way to say this is that R is self-adjoint on $L^2(\mathbb{Q})$. We remark that the adjoint R^* is the transition kernel for the time-reversed process with $\mathbb{Q}$ (normalised to a probability measure) as the initial distribution.

We can compute an invariant measure for some of the models we are studying in terms of $\mathbb{P}$.

We start with the bond model.

Example 2. Recall the invariant measure π for random walks among fixed conductances, given by

$$\pi(x) \propto \sum_{y \sim x} c(\{x, y\}) \quad (\text{not finite in general on infinite graphs}).$$

This makes sense as if the total conductance incident to x is larger, it is easier for a walk to reach it.

This inspires the following condition for the invariant measure for the EVP

$$d\mathbb{Q}(\omega) \propto \sum_{u \in S} \omega(\{0, u\}) d\mathbb{P}(\omega),$$

i.e. $d\mathbb{Q}(\omega)$ scales with sum of conductances around the origin, which can be interpreted as the likelihood of the particle to be at the origin of ω .

We are fortunate to be able to normalise this

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega) &\stackrel{\text{def}}{=} \frac{1}{E^{\mathbb{P}}[\sum_{u \in S} \omega(\{0, u\})]} \sum_{u \in S} \omega(\{0, u\}) \\ &= \frac{1}{Z} \sum_{u \in S} \omega(\{0, u\}), \end{aligned}$$

where $Z = E^{\mathbb{P}} [\sum_{u \in S} \omega(\{0, u\})]$. In fact, the EVP on the bond model is also reversible, which we now show.

For bounded measurable g, h on Ω , we have

$$\begin{aligned} \int h(Rg) d\mathbb{Q} &= \int h(\omega) \sum_{u \in S} p^\omega(0, u) g(t_u \omega) \mathbb{Q}(d\omega) \\ &= \int h(\omega) \sum_{u \in S} \frac{\omega(\{0, u\})}{\sum_{v \in S} \omega(\{0, v\})} g(t_u \omega) \cdot \frac{1}{Z} \sum_{u \in S} \omega(\{0, u\}) \mathbb{P}(d\omega) \\ &= \frac{1}{Z} \sum_{u \in S} \int h(\omega) \omega(\{0, u\}) g(t_u \omega) \mathbb{P}(d\omega). \end{aligned}$$

Since $\mathbb{P}$ is t_u -invariant, $E^{\mathbb{P}}[f] = E^{\mathbb{P}}[f \circ t_u^{-1}]$. Hence, we precompose the integrand with t_{-u} .

$$\begin{aligned} &= \frac{1}{Z} \sum_{u \in S} \int h(t_{-u} \omega) (t_{-u} \omega)(\{0, u\}) g(\omega) \mathbb{P}(d\omega) \\ &= \frac{1}{Z} \int g(\omega) \sum_{u \in S} h(t_{-u} \omega) \omega(\{-u, 0\}) \mathbb{P}(d\omega) \\ &= \frac{1}{Z} \int g(\omega) \sum_{u \in S} \omega(\{0, u\}) h(t_u \omega) \mathbb{P}(d\omega) \\ &= \int (Rh)g d\mathbb{Q}. \end{aligned}$$

We have shown that the EVP on the bond model admits an invariant probability measure and, furthermore, it is reversible.

We move onto the site model.

Example 3. Intuitively, in general, site models do not give rise to reversible EVPs, because there is some sort of asymmetry. Now, we try to discover the expression for the invariant probability measure $\mathbb{Q}$ in $d = 1$, with $E^{\mathbb{P}}[\rho_0] < 1$ (right drift, ρ_x defined below). The following derivation is similar to that in Molchanov, as cited by Sznitman.

Set $p_x = \omega(x, +1)$, $q_x = \omega(x, -1) = 1 - p_x$,

$$\rho_x = \frac{q_x}{p_x} = p_x^{-1} - 1.$$

We hope to find a density $f(\omega)$ s.t. $d\mathbb{Q} = f d\mathbb{P}$.

By definition, the invariance of $\mathbb{Q}$ is equivalent to that

$$\int (Rg)(\omega) f(\omega) \mathbb{P}(d\omega) = \int g(\omega) f(\omega) \mathbb{P}(d\omega),$$

for every bounded measurable g .

Applying the definition of R ,

$$\begin{aligned} \text{LHS} &= \int p_0(\omega) g(t_1 \omega) f(\omega) \mathbb{P}(d\omega) + \int q_0(\omega) g(t_{-1} \omega) f(\omega) \mathbb{P}(d\omega) \\ &= \int p_{-1} g(\omega) f(t_{-1} \omega) \mathbb{P}(d\omega) + \int q_1 g(\omega) f(t_1 \omega) \mathbb{P}(d\omega). \end{aligned}$$

If g bounded measurable, then $\forall x \in \mathbb{Z}$, $g \circ t_x$ also bounded measurable. This allows invariance to be equivalent to

$$\begin{aligned} \int p_{-1}(\omega)g(t_{-x}\omega)f(t_{-1}\omega)\mathbb{P}(d\omega) + \int q_1g(t_{-x}\omega)f(t_1\omega)\mathbb{P}(d\omega) \\ = \int g(t_{-x}\omega)f(\omega)\mathbb{P}(d\omega) \end{aligned}$$

for every $x \in \mathbb{Z}$, g bounded measurable. Translation invariance gives

$$\begin{aligned} \int p_{x-1}(\omega)g(\omega)f(t_{x-1}\omega)\mathbb{P}(d\omega) + \int q_{x+1}(\omega)g(\omega)f(t_{x+1}\omega)\mathbb{P}(d\omega) \\ = \int g(\omega)f(t_x\omega)\mathbb{P}(d\omega). \end{aligned}$$

Let $f_x \stackrel{\text{def}}{=} f(t_x\omega)$. We are saying that invariance is equivalent to

$$\int (p_{x-1}f_{x-1} + q_{x+1}f_{x+1})g \, d\mathbb{P} = \int f_x g \, d\mathbb{P},$$

for every bounded measurable g . Hence, we need $p_{x-1}f_{x-1} + q_{x+1}f_{x+1} = f_x$. Rearrange:

$$\begin{aligned} p_{x-1}f_{x-1} + q_{x+1}f_{x+1} &= p_x f_x + q_x f_x, \\ p_x f_x - q_{x+1}f_{x+1} &= p_{x-1}f_{x-1} - q_x f_x. \end{aligned}$$

This is equivalent to the expression being constant along x . We have an infinite recurrence relation with no boundary condition.

But we also want f to be a density, i.e. $E^{\mathbb{P}}[f] = 1$. In fact, this will determine a constant (in x , but not in ω , *a priori*)

$$a(\omega) = p_x f_x - q_{x+1} f_{x+1}$$

such that this normalisation condition is fulfilled. It is in fact almost surely constant in ω by ergodicity, since $a \circ t_x = a$. Perhaps this will give us a unique solution.

$$\begin{aligned} p_x f_x &= a + q_{x+1} f_{x+1} \\ &= a + \rho_{x+1} p_{x+1} f_{x+1} \\ &= a(1 + \rho_{x+1}) + \rho_{x+1} \rho_{x+2} p_{x+2} f_{x+2} \\ &= a \left(1 + \rho_{x+1} + \rho_{x+1} \rho_{x+2} + \cdots + \prod_{k=1}^n \rho_{x+k} \right) \\ &\quad + \left(\prod_{k=1}^{n+1} \rho_{x+k} \right) p_{x+n+1} f_{x+n+1}. \end{aligned}$$

Morally, this last term should go to zero as $n \rightarrow \infty$. Suppose for now that this is the case, putting $x = 0$ into the above yields

$$f_0 = \frac{1}{p_0} a(1 + \rho_1 + \rho_1 \rho_2 + \cdots).$$

Integrating,

$$E^{\mathbb{P}}[f_0] = aE^{\mathbb{P}} \left[\frac{1}{p_0} (1 + \rho_1 + \rho_1\rho_2 + \cdots) \right],$$

$$1 = aE^{\mathbb{P}}[p_0^{-1}] \cdot \frac{1}{1 - E^{\mathbb{P}}[\rho_0]} \quad (\rho_x \text{ iid}).$$

Since $p_0^{-1} = 1 + \rho_0$, we find that

$$a = \frac{1 - E^{\mathbb{P}}[\rho_0]}{1 + E^{\mathbb{P}}[\rho_0]}.$$

The desired density is thus

$$f(\omega) = \frac{1 - E^{\mathbb{P}}[\rho_0]}{1 + E^{\mathbb{P}}[\rho_0]} (1 + \rho_0(\omega))(1 + \rho_1(\omega) + \rho_1(\omega)\rho_2(\omega) + \cdots).$$

Back to the boundary term, we show it goes to zero. By the strong law,

$$\frac{1}{n} \sum_{k=1}^n \log(\rho_k) \xrightarrow[\mathbb{P}\text{-a.s.}]{n \rightarrow \infty} E^{\mathbb{P}}[\log(\rho_0)] \quad (< 0 \text{ by Jensen}).$$

So

$$\prod_{k=1}^n \rho_k = \exp \left(-n \cdot |E^{\mathbb{P}}[\log(\rho_0)]| + o(n) \right)$$

as $n \rightarrow +\infty$, $\mathbb{P}$ -a.s.

We always have $p_{x+n+1} \leq 1$, since it is a probability.

We want to show that f_{x+n+1} at least grows slow enough for the boundary term to vanish. By Markov's inequality, for any $\varepsilon > 0$,

$$\mathbb{P}[f(t_{x+n}\omega) > e^{\varepsilon n}] \leq \frac{E^{\mathbb{P}}[f(t_{x+n}\omega)]}{e^{\varepsilon n}} = e^{-\varepsilon n}.$$

$$\sum_{n=1}^{+\infty} \mathbb{P}[f(t_{x+n}\omega) > e^{\varepsilon n}] \leq \sum_{n=1}^{+\infty} e^{-\varepsilon n} < \infty.$$

So by Borel–Cantelli, $\mathbb{P}$ -a.s., $\exists N(\omega)$ s.t.

$$\forall n \geq N, \quad f(t_{x+n}\omega) \leq e^{\varepsilon n}.$$

Hence, we can choose ε sufficiently small, say $\varepsilon = \frac{1}{2}|E^{\mathbb{P}}[\log \rho_0]|$, so that the growth of f_{x+n+1} is curbed by the $\prod_{k=1}^{n+1} \rho_{x+k}$ term.

Hopefully, the above is a decent illustration for finding an expression for the density f . One should also check that it defines a valid invariant measure, which we omit here.

This concludes our computation of explicit invariant measures. We will make use of the above result later.

3 Ergodicity

We have seen invariant measures $\mathbb{Q} \ll \mathbb{P}$. This means that the EVP is not too different from sampling according to $\mathbb{P}$. Sznitman calls this a key assumption for models, that

The static POV and dynamic POV are comparable.

This says that we should have $\mathbb{P} \sim \mathbb{Q}$. We will now show this, and more. This is Sznitman's Theorem 1.1, stated in a more general way, although the proof is identical.

Theorem 1. Let $\mathbb{P}$ be an environment measure which is invariant and ergodic under translations $(t_x)_{x \in \mathbb{Z}^d}$, with $p^\omega(0, u) > 0$ for every $u \in S$, $\mathbb{P}$ -a.s. If there exists an invariant probability measure $\mathbb{Q}$ for the transition kernel R , $\mathbb{Q} \ll \mathbb{P}$, then

1. $\mathbb{Q} \sim \mathbb{P}$.
2. $\mathbb{Q}$ is ergodic for R , i.e. the system $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}_\mathbb{Q}, \tilde{\theta})$ is ergodic.
3. $\mathbb{Q}$ is the unique probability measure invariant for R with $\mathbb{Q} \ll \mathbb{P}$.

We first show a measure-theoretic lemma.

Lemma 1. For measures μ, ν with $d\nu = f d\mu$, $\mu[\{f = 0\}] = 0$ implies $\mu \ll \nu$.

Proof. Let A such that $\nu[A] = 0$. Then $0 = \nu[A] = \int_A f d\mu$. So $\mu[A \cap \{f > 0\}] = 0$. But $\{f > 0\}$ has full μ -measure, so $\mu[A] = 0$. $\square$

Proof of $\mathbb{Q} \sim \mathbb{P}$. We have already assumed $\mathbb{Q} \ll \mathbb{P}$, now we just need to show $\mathbb{P} \ll \mathbb{Q}$. Let $f = \frac{d\mathbb{Q}}{d\mathbb{P}}$.

Let $E = \{f = 0\}$. We will show that $\mathbb{P}[E] = 0$. We will do so by constructing a translation-invariant $\mathbb{P}$ -version $\tilde{E}$ of E , from which we deduce by ergodicity of $\mathbb{P}$ that $\mathbb{P}[\tilde{E}] \in \{0, 1\}$. But if $\mathbb{P}[\tilde{E}] = 1$, then $\mathbb{P}[E] = 1$ and $\int f d\mathbb{P} = 0$, which is a contradiction.

We define $\tilde{E} = \bigcap_{x \in \mathbb{Z}^d} t_x^{-1}(E)$. This is clearly translation invariant. Is it a $\mathbb{P}$ -version of E ?

In fact, we now show that every $t_x^{-1}(E)$ is a $\mathbb{P}$ -version of E , so $\tilde{E}$ is a $\mathbb{P}$ -version by countability of $\mathbb{Z}^d$. We want to study relationship between E and its translations. Most directly, we can consider $R1_E$.

Invariance of $\mathbb{Q}$ gives

$$\int R1_E d\mathbb{Q} = \int 1_E d\mathbb{Q} = \int_{\{f=0\}} f d\mathbb{P} = 0.$$

$$\int R1_E f d\mathbb{P} = 0.$$

In particular, $R1_E = 0$ $\mathbb{P}$ -a.s. on $\{f > 0\} = E^c$. On E , we have $1_E = 1 \geq R1_E$. Putting these together, $R1_E \leq 1_E$ $\mathbb{P}$ -a.s.

$$\sum_{u \in S} p^\omega(0, u) \underbrace{1_E(t_u \omega)}_{=1_{t_u^{-1}E}(\omega)} \leq 1_E, \quad \mathbb{P}\text{-a.s.}$$

In particular,

$$\left(\mathbf{1}_E \geq \underbrace{p^\omega(0, u)}_{>0} \mathbf{1}_{t_u^{-1}E} \iff \mathbf{1}_E \geq \mathbf{1}_{t_u^{-1}E} \right) \quad \mathbb{P}\text{-a.s.}$$

Hence, $t_u^{-1}E \subseteq E$ $\mathbb{P}$ -a.s. But by translation invariance, $\mathbb{P}[E] = \mathbb{P}[t_u^{-1}E]$. So $t_u^{-1}E$ is a $\mathbb{P}$ -version of E . This is true for all $u \in S$, so iteratively, $t_x^{-1}E$ is a $\mathbb{P}$ -version of E , which is what we wanted. $\square$

We can now write $\mathbb{Q}$ -a.s. and $\mathbb{P}$ -a.s. interchangeably, which we will need below. First, we recall an important theorem, which we will also appeal to frequently.

Lemma 2 (Birkhoff's ergodic theorem). Let $(\Omega, \mathcal{F}, \mathbb{P}, \tau)$ be a measure-preserving dynamical system and $X \in L^1(\mathbb{P})$. Then

$$\frac{1}{n} \sum_{k=0}^{n-1} X \circ \tau^k \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-a.s.}} E[X \mid \mathcal{I}],$$

where $\mathcal{I}$ is the σ -algebra of τ -invariant events. If τ is ergodic, then $E[X \mid \mathcal{I}] = E[X]$.

Proof of $\mathbb{Q}$ ergodic for R . Let $A \in \widetilde{\mathcal{F}}$, s.t. $\tilde{\theta}^{-1}(A) = A$, be a shift invariant event. We want to show that $\tilde{P}_{\mathbb{Q}}[A] \in \{0, 1\}$. Note that

$$\tilde{P}_{\mathbb{Q}}[A] = \int \tilde{P}_{\omega}[A] \mathbb{Q}(d\omega),$$

so what we really want to do is to show that $\tilde{P}_{\omega}[A]$ is constant (either 0 or 1) $\mathbb{Q}$ -a.s. on Ω . That is, we want to find some set $B \in \mathcal{F}$ with $\mathbb{Q}$ -measure one or zero s.t. $\tilde{P}_{\omega}[A] = \mathbf{1}_B$ $\mathbb{Q}$ -a.s.

Let $\varphi(\omega) = \tilde{P}_{\omega}[A]$ for brevity.

In fact, $\varphi(\tilde{\omega}_n)$ is a $\tilde{P}_{\mathbb{Q}}$ -martingale, for $\widetilde{\mathcal{F}}_n$. By $\tilde{\theta}$ -invariance of A ,

$$\begin{aligned} \tilde{E}_{\mathbb{Q}}[\mathbf{1}_A \mid \widetilde{\mathcal{F}}_n] &= \tilde{E}_{\mathbb{Q}}[\mathbf{1}_A \circ \tilde{\theta}^n \mid \widetilde{\mathcal{F}}_n] \\ &= \tilde{P}_{\tilde{\omega}_n}[A] = \varphi(\tilde{\omega}_n). \end{aligned}$$

This is a Lévy-martingale, so we apply Lévy's upward theorem (after noting that $\widetilde{\mathcal{F}} = \sigma(\bigcup_{n \geq 0} \widetilde{\mathcal{F}}_n)$), to yield

$$\varphi(\tilde{\omega}_n) \longrightarrow \mathbf{1}_A \quad \text{as } n \rightarrow +\infty, \quad \tilde{P}_{\mathbb{Q}}\text{-a.s.}$$

We now show that $\varphi = \mathbf{1}_B$ $\mathbb{Q}$ -a.s. for some $B \in \mathcal{F}$. (Why don't I just drop the $\mathbb{Q}$ -a.s. and using a different version of B instead? Well, we cannot exclude other values in the range of φ .)

Suppose this is not the case, i.e. the event $\{\varphi \in (0, 1)\}$ has positive $\mathbb{Q}$ -probability. Then, in fact, $\exists [a, b] \subseteq (0, 1)$ s.t. $\mathbb{Q}[\varphi \in [a, b]] > 0$. After checking that $\tilde{P}_{\mathbb{Q}} \circ \tilde{\theta}^{-1} = \tilde{P}_{\mathbb{Q}}$ from $\mathbb{Q}R = \mathbb{Q}$, we apply Birkhoff's ergodic theorem to find that,

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\{\varphi(\tilde{\omega}_k) \in [a, b]\}} \longrightarrow \tilde{E}_{\mathbb{Q}}[\mathbf{1}_{\{\varphi(\tilde{\omega}_0) \in [a, b]\}} \mid \mathcal{I}] \quad \tilde{P}_{\mathbb{Q}}\text{-a.s.}$$

$$\begin{aligned}
\tilde{E}_{\mathbb{Q}}[\tilde{E}_{\mathbb{Q}}[\mathbf{1}_{\{\varphi(\tilde{\omega}_0) \in [a,b]\}} \mid \mathcal{I}]] &= \tilde{P}_{\mathbb{Q}}[\varphi(\tilde{\omega}_0) \in [a,b]] \\
&= \mathbb{Q}[\varphi \in [a,b]] \\
&> 0.
\end{aligned}$$

But since $\varphi(\tilde{\omega}_n) \rightarrow \mathbf{1}_A$, $\tilde{P}_{\mathbb{Q}}$ -a.s., along almost every trajectory $\tilde{\omega}$, $\varphi(\tilde{\omega}_n) \rightarrow 1$ or 0. So we really should have

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\{\varphi(\tilde{\omega}_k) \in [a,b]\}} \longrightarrow 0 \quad \tilde{P}_{\mathbb{Q}}\text{-a.s.}$$

But the above says this converges to a non-negative random variable with positive expectation. We have arrived at a contradiction.

So $\varphi = \mathbf{1}_B$ $\mathbb{Q}$ -a.s. for some $B \in \mathcal{F}$. Now we are going to show that $\mathbb{Q}[B] \in \{0, 1\}$. This now amounts to showing $\mathbb{P}[B] \in \{0, 1\}$. We have just done something like this a while ago, in the proof of $\mathbb{P} \ll \mathbb{Q}$. We will do the same here. What was really fundamental in that proof was $R\mathbf{1}_E \leq \mathbf{1}_E$, $\mathbb{P}$ -a.s. We have something better.

$$\mathbf{1}_B(\tilde{\omega}_0) = \varphi(\tilde{\omega}_0) = \tilde{E}_{\mathbb{Q}}[\varphi(\tilde{\omega}_1) \mid \tilde{\mathcal{F}}_0] \stackrel{\text{stationarity}}{=} \tilde{E}_{\mathbb{Q}}[\mathbf{1}_B(\tilde{\omega}_1) \mid \tilde{\mathcal{F}}_0] \stackrel{\text{Markov}}{=} (R\mathbf{1}_B)(\tilde{\omega}_0) \quad \tilde{P}_{\mathbb{Q}}\text{-a.s.},$$

which implies $\mathbf{1}_B(\omega) = (R\mathbf{1}_B)(\omega)$ for $\mathbb{Q}$ -a.e. ω . We applied stationarity of $\tilde{P}_{\mathbb{Q}}$, that $\tilde{\omega}_1$ has law $\mathbb{Q}$, to show that $\varphi(\tilde{\omega}_1) = \mathbf{1}_B(\tilde{\omega}_1)$, $\tilde{P}_{\mathbb{Q}}$ -a.s. Hence, $\mathbf{1}_B = R\mathbf{1}_B$ $\mathbb{Q}$ -a.s., and therefore $\mathbb{P}$ -a.s. We reapply the argument to obtain $\mathbb{P}[B] \in \{0, 1\}$, which is what we wanted. $\square$

Proof of uniqueness of $\mathbb{Q}$. Let $\mathbb{Q}' \ll \mathbb{P}$ be another invariant probability measure for R . Then $\mathbb{Q}' \sim \mathbb{P}$ and the chain under $\tilde{P}_{\mathbb{Q}'}$ is ergodic.

Then for any bounded measurable $g : \Omega \rightarrow \mathbb{R}$, the ergodic theorem gives

$$\frac{1}{n} \sum_{k=0}^{n-1} g(\tilde{\omega}_k) \longrightarrow \int g \, d\mathbb{Q} \quad \tilde{P}_{\mathbb{Q}}\text{-a.s.}$$

and

$$\frac{1}{n} \sum_{k=0}^{n-1} g(\tilde{\omega}_k) \longrightarrow \int g \, d\mathbb{Q}' \quad \tilde{P}_{\mathbb{Q}'}\text{-a.s.}$$

To show that $\int g \, d\mathbb{Q} = \int g \, d\mathbb{Q}'$, what we really need is that for each g , there exists a trajectory $\tilde{\omega}$ such that both of the convergences hold.

From $\mathbb{Q} \sim \mathbb{P} \sim \mathbb{Q}'$, we can show that $\tilde{P}_{\mathbb{Q}} \sim \tilde{P}_{\mathbb{P}} \sim \tilde{P}_{\mathbb{Q}'}$. So we in fact have a $\tilde{P}_{\mathbb{P}}/\tilde{P}_{\mathbb{Q}}/\tilde{P}_{\mathbb{Q}'}$ -measure one set of trajectories on which both convergences hold. $\square$