

Errata and proof corrections

Selected Notes on Schramm–Loewner Evolutions

Errata by ChatGPT (OpenAI)

Notes by Baining Zhang, dated 29 April 2025. Errata updated 5 October 2026. Source: [slenotes.pdf](#).

These errata accompany the 29 April 2025 version of the notes. Page numbers and theorem, proposition, lemma, and definition numbers refer to those notes unless another source is named. Entries E01–E29 give corrected statements and supporting arguments; the final sections collect smaller corrections and clarify conventions. References [R1]–[R6] are listed at the end.

The entries most relevant to later arguments are **E11** (distortion proof), **E15** (Loewner initial condition), **E16–E19** (Brownian filtrations and stopping times), **E20–E22** (Loewner continuity and topology), and **E23, E25, E29** (SLE hypotheses and natural time).

Riemann maps and univalent functions

E01. Surjectivity of normalized conformal maps

Location: pp. 2–5; also Corollary 1 on p. 10

Injective holomorphic maps need not be onto. In the conformal-radius discussion, Propositions 2 and 3, and the normalized mapping theorems, the relevant maps are *conformal isomorphisms*: their stated target domains are their images.

For example, $F_1(z) = z$ and $F_2(z) = z/2$ are both univalent maps $\mathbb{D} \rightarrow \mathbb{D}$ fixing zero, but they are not related by a rotation and have different derivative moduli at zero. Thus the arbitrary-map definition of conformal radius and Proposition 2 are false literally as written. In Proposition 3, taking these maps with $\Omega_1 = \Omega_2 = \mathbb{D}$ even reverses the claimed inequality.

The conformal-radius assumptions are $z_0 \in \Omega \subsetneq \mathbb{C}$, with Ω a nonempty simply connected domain, and $F : \mathbb{D} \xrightarrow{\sim} \Omega$. In Corollary 1, $\Omega = f(\mathbb{D})$.

E02. Extremal characterization of the conformal radius

Location: p. 2, proof of Proposition 1

Among holomorphic maps $F : \mathbb{D} \rightarrow \Omega$ with $F(0) = z_0$, the Riemann map maximizes $|F'(0)|$; it does not minimize it. Precomposing a Riemann map with $z \mapsto rz$, $0 < r < 1$, makes the derivative smaller.

The corrected extremal characterization has the following proof. Let $F : \mathbb{D} \xrightarrow{\sim} \Omega$ have $F(0) = z_0$ and $F'(0) = \rho > 0$. If f is one of the normalized univalent maps on Ω and $M = \sup_{\Omega} |f| < \infty$, Schwarz’s lemma applied to $f \circ F/M$ gives

$$\rho/M \leq 1, \quad M \geq \rho.$$

The choice $f = \rho F^{-1}$ attains equality. The equality case of Schwarz’s lemma, together with $f'(z_0) = 1$, proves uniqueness. The supremum is taken over $z \in \Omega$, rather than $z \in \mathbb{D}$.

E03. Cayley transform

Location: p. 3, last display

The displayed map $(i - z)/(i + z)$ maps the disc onto the *right* half-plane. In particular, it sends 0 to 1, which is not in $\mathbb{H}$. The corrected map, with its pole at $-i$, is

$$\phi(z) = i \frac{i - z}{i + z}.$$

Then $\phi : \mathbb{D} \xrightarrow{\sim} \mathbb{H}$ and $\phi(-i) = \infty$.

E04. Boundary extension near infinity

Location: pp. 3–4, proof of Theorem 1

The assumed continuous injective extension to all of $\mathbb{R} \cap \partial\Omega^+$ is not justified for an arbitrary hull. Nor does reflection across that entire set automatically give an open domain on which the extension is holomorphic.

For R with $A \subset \{|z| < R\}$, boundary correspondence on the analytic boundary arc through infinity allows reflection across the real rays outside this disc. This provides the extension near infinity needed for the Laurent expansion. Values at attachment points of the hull need not be defined for this argument. Also, the reflected map $\hat{g}$ does not map its lower-half-plane portion into $\mathbb{H}$; the assertion about mapping into $\mathbb{H}$ holds for its restriction to Ω^+ .

E05. Reality of the Laurent coefficients

Location: p. 4, argument that the Laurent coefficients are real

In general,

$$\hat{g}(R) - a_1 R \longrightarrow a_0,$$

so the displayed equality asserting that its modulus tends to zero is false. Instead,

$$a_1 = \lim_{R \rightarrow +\infty} \frac{\hat{g}(R)}{R} \in \mathbb{R}.$$

The reflection identity $\hat{g}(\bar{z}) = \overline{\hat{g}(z)}$, together with uniqueness of Laurent coefficients, gives reality of every coefficient. Since g maps the upper half-plane near infinity into $\mathbb{H}$, $a_1 > 0$. The subsequent normalization is then valid.

E06. Composition in the uniqueness argument

Location: p. 5, uniqueness part of Theorem 1

The correct implication is

$$g_1 \circ g_2^{-1}(w) = \alpha w + \beta \implies g_1(z) = \alpha g_2(z) + \beta.$$

It does not imply $g_1(z) = g_2(\alpha z + \beta)$. Substituting $g_j(z) = z + O(z^{-1})$ into the correct identity gives $\alpha = 1$, $\beta = 0$ immediately.

E07. Nondegenerate complements and translation

Location: p. 5, Theorem 2 and its proof

The theorem requires K to contain at least two points. A singleton K satisfies the stated simply-connected-complement condition on the sphere, but its complement is conformally the plane and cannot be mapped conformally from the exterior disc.

For $z_0 \in K$, the correct translation formula is

$$F_K(z) = z_0 + F_{K-z_0}(z),$$

not a translation of the argument. The domain of F_K is the fixed exterior unit disc. Here $\mathbb{D}_-$ denotes the exterior unit disc, including infinity when regarded as a domain on the Riemann sphere. The normalization at infinity means $\lim_{z \rightarrow \infty} F_K(z)/z = a_1 > 0$.

E08. Contour integration in the area theorem

Location: p. 7, proof of the area theorem

The first area integral is over the image curve:

$$\lambda_2(K_r) = \frac{1}{2i} \int_{g(r\partial\mathbb{D})} \bar{w} dw = \frac{1}{2i} \int_{r\partial\mathbb{D}} \overline{g(z)} g'(z) dz.$$

The displayed bound for the negative-power tail has the powers of r inverted and omits the contour length. A valid bound is

$$\left| \int_{r\partial\mathbb{D}} \sum_{n>N} n b_n z^{-n-2} dz \right| \leq 2\pi r \sum_{n>N} n |b_n| r^{-n-2} \rightarrow 0.$$

Uniform absolute convergence on $r\partial\mathbb{D}$ justifies the termwise integration. The final area formula is correct. The opening claim that a merely continuous boundary extension would permit a direct use of Green's theorem also needs boundary regularity; the $r > 1$ argument avoids that issue.

E09. Equality case in Bieberbach's estimate

Location: pp. 9–10, Proposition 5

The equality case is

$$|a_2| = 2 \iff f(z) = e^{-i\theta} f_{\text{Koebe}}(e^{i\theta} z) = \frac{z}{(1 - e^{i\theta} z)^2} \text{ for some } \theta \in \mathbb{R}.$$

Multiplication by $e^{i\theta}$ alone generally destroys $f'(0) = 1$. This is the form already appearing in the last line of the proof on p. 10; it applies to both the statement and the earlier direction of the proof.

For the auxiliary exterior map $g(z) = z + \sum_{n \geq 1} c_n z^{-n}$, the area-theorem estimate is

$$1 \geq \sum_{n \geq 1} n |c_n|^2 \geq |c_1|^2 = |a_2|^2 / 4 = 1.$$

Consequently $c_n = 0$ for $n \geq 2$. The printed sum without squares is not the area-theorem estimate.

E10. Normalization in the quarter-theorem corollary

Location: p. 11, proof of Corollary 2

Write $a = \tilde{f}'(0) = \rho(0, \Omega)$ and $f = \tilde{f}/a$. Then $f'(0) = 1$, while $\tilde{f}'(0)$ need not be 1. The correct chain is

$$\frac{1}{4} \leq \text{dist}(0, \partial f(\mathbb{D})) = \frac{\text{dist}(0, \partial\Omega)}{a} \leq 1.$$

This proves $1 \leq \rho(0, \Omega) / \text{dist}(0, \partial\Omega) \leq 4$. The printed proof switches f and $\tilde{f}$, uses an undefined F , and applies the distance bound to the wrong image.

E11. Conformal-radius factors in the distortion estimate

Location: pp. 11–12, proof of Theorem 6

For conformal isomorphisms $g : \mathbb{D} \rightarrow \Omega$, $h : \mathbb{D} \rightarrow \Omega'$, and $\zeta \in \mathbb{D}$,

$$\rho(g(\zeta), \Omega) = (1 - |\zeta|^2) |g'(\zeta)|, \quad \rho(h(\zeta), \Omega') = (1 - |\zeta|^2) |h'(\zeta)|.$$

The two estimates on p. 12 each contain the factor $(1 - |\zeta|^2)$; that factor cancels when they are multiplied. For instance, with $h = \text{id}_{\mathbb{D}}$ and $\zeta = 0.9$, the printed upper bound would assert $10 \leq 4$.

The decomposition through $\mathbb{D}$ also presupposes proper simply connected domains. The *conclusion* actually holds for conformal isomorphisms between arbitrary proper plane domains: apply Koebe's quarter theorem to the restriction of f to $B(z, \text{dist}(z, \partial\Omega))$, then to the corresponding disc for f^{-1} . This yields the two inequalities without a simply-connectedness assumption. Surjectivity onto Ω' is still necessary.

Capacity and deterministic Loewner theory

E12. Translation and scaling of exterior maps

Location: p. 12, immediately after Definition 5

The transformation rules are

$$F_{K+w}(z) = F_K(z) + w, \quad F_{aK}(z) = aF_K(z) \quad (a > 0).$$

The printed $F_{K+w} = F_K$ is false. The claimed translation invariance of cap nevertheless follows, since the coefficient of z is unchanged. The scaling formula involves $\text{cap}(K)$, where $\text{cap}(A)$ is printed.

E13. Schwarz's lemma for the comparison map

Location: p. 13, proof of Proposition 6

From $K_1 \subseteq K_2$, one obtains

$$f_{K_1}^{-1} \circ f_{K_2} : \mathbb{D} \longrightarrow \mathbb{D}.$$

This is generally a proper univalent self-map, not an element of $\text{Aut}(\mathbb{D})$. Schwarz's lemma applies to a self-map and proves the desired inequality. Only in the equality case does the composition become a rotation (and, under the positive-derivative normalization, the identity).

E14. Closed hulls and restart maps

Location: pp. 13–14 and restart formulas on pp. 23–25

Definition 1 uses compact subsets of $\overline{\mathbb{H}}$ satisfying $A = \overline{A} \cap \overline{\mathbb{H}}$. Accordingly the examples on p. 14 are

$$A = \overline{\mathbb{D}} \cap \overline{\mathbb{H}}, \quad A' = [0, i].$$

Their capacities are 1 and $1/2$, respectively, and the mapping-out formulas are correct with the branch asymptotic to z at infinity. The capacity statement is $\text{hcap}(A) = 1$; the whole closed disc is not a half-plane hull.

The closures already printed in Theorem 12 are appropriate. A precise restart definition, restricting the argument of g_s to its domain, is

$$\widehat{K}_{s,t} = \overline{g_s((K_{s+t} \setminus K_s) \cap \mathbb{H}) - W_s}.$$

The same convention applies in the proof of Theorem 13. On p. 25, applying g_s to all of K_{s+t} is invalid because g_s is not defined on K_s .

For the capacity-addition argument, set $B_A = \overline{g_A(B \cap \mathbb{H})}$ for disjoint closed hulls; the boundary extension away from A makes this well defined. Then $g_{A \cup B} = g_{B_A} \circ g_A$ and the stated capacity identity follows.

E15. Initial condition and local growth

Location: p. 15, Theorem 7

The differential-equation condition must include $g_0(z) = z$, equivalently $K_0 = \emptyset$, with each solution considered up to its swallowing time. Without it,

$$g_t(z) = \sqrt{z^2 + 4(t+a)}, \quad a > 0,$$

satisfies the printed equation with $W_t = 0$, but its hull has capacity $2(t+a)$ rather than $2t$.

In the local-growth condition, the undefined K means K_s . The condition concerns *all sufficiently short intervals*: for every $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $0 \leq s < t \leq T$ and $t - s < \delta$, a connected set in $\mathbb{H} \setminus K_s$ of diameter below ε separates the new hull from infinity. With the closed-hull convention, this means the part of the new hull in $\mathbb{H} \setminus K_s$. The printed quantifier permits $\delta > T$, making its separation condition vacuous.

Brownian motion and stopping times

E16. Independence in the limiting argument

Location: p. 16, proof of Theorem 9

The limiting increment is not generally independent of each $\mathcal{F}^0(s_n)$. For $s = 0$ and $0 < t < s_n$, for instance, $B_t - B_0$ is itself $\mathcal{F}^0(s_n)$ -measurable and nonconstant.

The correct argument uses

$$\mathcal{F}^+(s) \subseteq \mathcal{F}^0(s_n) \quad \text{for every } n.$$

Every shifted future process is therefore independent of the *same* sigma-algebra $\mathcal{F}^+(s)$. Its finite-dimensional vectors converge almost surely by path continuity. Passing to the limit against bounded continuous test functions preserves the independence factorization. A monotone-class argument gives independence of the entire limiting process. Merely deleting finitely many terms does not prove independence from $\mathcal{F}^0(s_n)$.

E17. Hitting-time event formulas

Location: p. 17, the two hitting-time displays

The printed closed-set formula is not valid: $\mathbb{Q}^d \cap F$ may be empty even when F is nonempty, and the displayed intersection over centers does not express proximity to F . A formula valid for continuous paths, including $t = 0$, is

$$\{\tilde{H}(F) \leq t\} = \bigcap_{n \geq 1} \bigcup_{q \in Q_t} \{\text{dist}(B_q, F) < 1/n\}, \quad Q_t = (\mathbb{Q} \cap [0, t]) \cup \{t\}.$$

Continuity and compactness of $[0, t]$ make the infimum distance attain its minimum; closedness of F then gives a hit.

For an open set G , the inner operation in the second display must be a *union*, not an intersection:

$$\{\tilde{H}(G) < s\} = \bigcup_{r \in \mathbb{Q} \cap [0, s)} \{B_r \in G\}, \quad s > 0,$$

and hence

$$\{\tilde{H}(G) \leq t\} = \bigcap_{m \geq 1} \bigcup_{r \in \mathbb{Q} \cap [0, t+1/m)} \{B_r \in G\} \in \mathcal{F}^+(t).$$

The membership in every $\mathcal{F}^0(t + 1/n)$ follows by using the tail $m \geq n$ of the outer intersection.

E18. Strict and non-strict stopping events

Location: p. 18, paragraph before Theorem 10

Right-continuity does not imply $\{T \leq t\} = \{T < t\}$. For the deterministic stopping time $T = 1$, these events are the whole sample space and the empty set at $t = 1$.

The valid statements are

$$\{T \leq t\} = \bigcap_{n \geq 1} \{T < t + 1/n\},$$

and, for a right-continuous filtration, the equivalence of the two *measurability criteria* for a stopping time. Equivalently one may define $\mathcal{F}(T)$ using either the conditions $A \cap \{T \leq t\} \in \mathcal{F}(t)$ for all t , or $A \cap \{T < t\} \in \mathcal{F}(t)$ for all t .

E19. Factorization at a stopping time

Location: pp. 18–19, proof of Theorem 10

For a Borel set A of continuous paths, $E \in \mathcal{F}^+(T_n)$, and ν the standard Wiener law, the required factorization is

$$\begin{aligned} \mathbb{P}(B_* \in A, E) &= \sum_{k \geq 0} \mathbb{P}(B_k \in A, E \cap \{T_n = k2^{-n}\}) \\ &= \sum_{k \geq 0} \nu(A) \mathbb{P}(E \cap \{T_n = k2^{-n}\}) \\ &= \nu(A) \mathbb{P}(E). \end{aligned}$$

The printed left-hand side omits E , and the next displayed line omits $\nu(A)$. If the original Brownian motion starts at $x \neq 0$, its unshifted path law cannot stand in for ν .

For the final limit, $\mathcal{F}^+(T) \subseteq \mathcal{F}^+(T_n)$, and the shifted paths converge almost surely, uniformly on each compact time interval. Applying bounded continuous path functionals to the preceding factorization with $E \in \mathcal{F}^+(T)$ and passing to the limit proves both Wiener law and independence.

Continuity of the Loewner transform

E20. Uniform control for forward Loewner maps

Location: p. 20, Theorem 11

Disjointness from G does not provide a uniform lower bound on $\Im g_t(z)$. A valid version fixes $T, \delta > 0$ and assumes

$$\inf_{z \in G} \Im g_k(T, z) \geq \delta, \quad k = 1, 2.$$

Lemma 2 then gives, for example,

$$\|g_1 - g_2\|_{[0, T] \times G} \leq \sqrt{1 + 4T/\delta^2} \|W_1 - W_2\|_{[0, T]}.$$

Here imaginary parts decrease with time, so the terminal lower bound controls the whole interval.

Comparison with Kemppainen. The same formulation without uniform geometric control occurs in Kemppainen [R4], p. 73, Section 5.1.2. However, his Proposition 6.2 on p. 104 has an additional hypothesis: a *fixed hull* K^0 contains both terminal hulls, and G is a compact subset of $\mathbb{H} \setminus K^0$. That proposition is correct.

Indeed, put $m = \min_{z \in G} \Im g_{K^0}(z) > 0$. If $A = K^k(t) \subseteq K^0$, the composition rule gives

$$g_{K^0} = g_B \circ g_A, \quad B = g_A(K^0 \setminus A).$$

Since a mapping-out function satisfies $\Im g_B(w) \leq \Im w$, we have

$$\Im g_k(t, z) \geq \Im g_{K^0}(z) \geq m \quad (t \leq T, z \in G).$$

Lemma 2 applies with, for example, $\delta = m/2$. The resulting constant depends on T, K^0, G and is uniform over the admissible chains; δ remains fixed.

Counterexample to the printed statement. Fix $T = 1$, $G = \{2i\}$, and compare constant drivers $W^{(a)} = a$ and $W^{(2a)} = 2a$, $a > 0$. Both vertical slits avoid G , while

$$g_1^{(a)}(2i) = a - \sqrt{a^2 - 4ia},$$

where the square root on the right is the principal one. Consequently

$$|g_1^{(2a)}(2i) - g_1^{(a)}(2i)| \sim 2(\sqrt{2} - 1)\sqrt{a}, \quad \|W^{(2a)} - W^{(a)}\| = a.$$

Their ratio diverges. Continuity at a fixed driving function on compacts of its surviving domain is still valid, by an ODE stability argument with a positive survival margin.

E21. Signed differences in the stability estimate

Location: pp. 21–22, proof of Lemma 2

Set $A_k(t) = g_k(t, z_k) - W_k(t)$, $\psi = g_1 - g_2$, and

$$D(t) = W_1(t) - W_2(t), \quad \zeta(t) = -\frac{2}{A_1(t)A_2(t)}.$$

Then

$$\psi'(t) = \frac{2(A_2 - A_1)}{A_1 A_2} = \zeta(t)(\psi(t) - D(t)).$$

The printed intermediate numerator has the opposite sign, and replacing D by $|W_1 - W_2|$ makes the equation false. Also $\psi(0) = z_1 - z_2$, not $|z_1 - z_2|$. Thus the correct exact solution is

$$\psi(t) = e^{\int_0^t \zeta(s) ds} (z_1 - z_2) - \int_0^t \zeta(u) e^{\int_u^t \zeta(s) ds} D(u) du.$$

The subsequent inequality follows by taking moduli of this identity.

On p. 22, the differential identity is

$$\partial_s \log \frac{\Im z_k}{\Im g_k(s, z_k)} = \frac{2}{|g_k(s, z_k) - W_k(s)|^2},$$

and the integrated expression, evaluated at t , is

$$I_k = \log \frac{\Im z_k}{\Im g_k(t, z_k)}.$$

With these repairs, the final constant $\sqrt{1 + 4T/\delta^2}$ is correct.

Derivation of the stability constant. The following calculation also completes the estimate in Kemppainen [R4], Lemma 6.2, p. 104. Write $Y_k(s) = \Im g_k(s, z_k)$ and $y_k = Y_k(0)$. The Loewner equation gives

$$(Y_k^2)' = -\frac{4Y_k^2}{|g_k - W_k|^2} \geq -4, \quad y_k^2 \leq Y_k(t)^2 + 4t.$$

Consequently, when $Y_k(t) \geq \delta$,

$$I_k = \log \frac{y_k}{Y_k(t)} \leq \frac{1}{2} \log \left(1 + \frac{4t}{Y_k(t)^2} \right) \leq \frac{1}{2} \log \left(1 + \frac{4T}{\delta^2} \right).$$

Cauchy–Schwarz bounds $\int_0^t |\zeta|$ by $\sqrt{I_1 I_2}$, so the integrating-factor formula yields the more precise estimate

$$|\psi(t)| \leq M|z_1 - z_2| + (M - 1)\|W_1 - W_2\|_{[0, T]}, \quad M = \sqrt{1 + 4T/\delta^2}.$$

Equivalently, the last ratio in Kemppainen’s hint is at most $\sqrt{1 + 4t/\delta^2}$, as follows by considering $y_k^2 \leq \delta^2 + 4t$ and $y_k^2 > \delta^2 + 4t$ separately. The estimate holds for each fixed $\delta > 0$; it does not assert uniform control as $\delta \downarrow 0$.

E22. The stated hull topology does not give a homeomorphism

Location: p. 23, paragraph invoking Loewner–Kufarev

For the topology of locally uniform convergence of normalized inverse maps, uniformly on compact time intervals, the map $L : W \mapsto (K_t)$ is a continuous bijection, but its inverse is not continuous when driving functions have the compact-open topology. Thus the homeomorphism assertion fails for these topologies. Continuity and measurability of L , which are what the construction of SLE needs here, remain valid.

Counterexample. Let $W = 0$ and define the continuous triangular spikes

$$W_n(t) = \begin{cases} nt, & 0 \leq t \leq 1/n, \\ 2 - nt, & 1/n \leq t \leq 2/n, \\ 0, & t \geq 2/n. \end{cases}$$

For every fixed $T > 0$ and all large n ,

$$\|W_n\|_{[0,T]} = 1, \quad \int_0^T |W_n(t)| dt = 1/n.$$

Let $f_t^{(n)} = (g_t^{(n)})^{-1}$ and $f_t = g_t^{-1}$. For terminal time t , the inverse map is obtained by solving the reverse ODE

$$\partial_s h_s = -\frac{2}{h_s - W(t-s)}, \quad h_0 = z, \quad f_t(z) = h_t.$$

Its imaginary part is increasing. For $\Im z \geq \delta > 0$, subtraction of two reverse equations and Gronwall's inequality give

$$\sup_{0 \leq t \leq T} |f_t^{(n)}(z) - f_t(z)| \leq \frac{2e^{2T/\delta^2}}{\delta^2} \int_0^T |W_n(u)| du \leq \frac{2e^{2T/\delta^2}}{n\delta^2}.$$

Thus the hull chains converge in the specified topology although their drivers do not converge uniformly. The same counterexample applies to the homeomorphism assertion in Berestycki–Norris [R1], Theorem 4.11, which uses this topology.

The Loewner correspondence. The correspondence between continuous drivers and increasing hull chains with the local growth property and half-plane capacity $2t$ is valid. See Kemppainen, Theorem 4.2, or Theorem 2.6 of Lawler–Schramm–Werner [R6] (their $A(K)$ equals $\text{hcap}(K)/2$). That representation theorem does not assert this homeomorphism. Kemppainen's Definition 6.2 uses the same inverse-map convergence as above; his Proposition 6.1 asserts continuity only from drivers to chains. His Theorem 6.2 concerns uniform convergence of *curves*, a stronger premise than convergence of hulls in the Carathéodory topology. The reverse-flow stability argument is also consistent with [R5], Lemma 2.3 and the preceding observation about L^1 perturbations.

SLE properties and changes of domain

E23. Brownian motion relative to the chosen filtration

Location: p. 23, Theorem 12

The hypothesis is that $B = W/\sqrt{\kappa}$ is Brownian motion *with respect to* $(\mathcal{F}_t)$. Adaptedness alone is insufficient. The usual augmented natural Brownian filtration, or the right-continuous natural filtration used earlier in the notes, satisfies the required condition.

As a counterexample to adaptedness alone, take $\mathcal{F}_t = \sigma(B_u : u \geq 0)$ for every t . Brownian motion is adapted, but future increments are already measurable at time zero and are not independent of $\mathcal{F}_0$. Accordingly the independence claims for the restarted SLE cannot hold for this filtration.

E24. Driving function after restarting

Location: p. 24, conformal Markov calculation

The restarted driver is

$$\widehat{W}_{s,t} = W_{s+t} - W_s.$$

In the denominator of the final differentiated expression and in the sentence immediately following it, the subtracted term is W_s , where W_t is printed. The later Brownian-increment display on the same page already has the correct expression.

E25. The deterministic case and drift scaling

Location: p. 25, Theorem 13 and its proof

The conclusion must allow $\kappa \geq 0$. The deterministic vertical slit driven by $W = 0$ satisfies scale invariance and the conformal Markov property, so these hypotheses do not imply $\kappa > 0$. The Lévy-process representation is

$$W_t = \sigma B_t + \alpha t, \quad \sigma \geq 0.$$

Continuity of the driver, together with stationary independent increments, excludes jumps. The Markov property at $s = 0$ also gives $W_0 = 0$ in law, hence almost surely.

The scaled drift is

$$\lambda W_{t/\lambda^2} = \lambda \sigma B_{t/\lambda^2} + \frac{\alpha t}{\lambda},$$

not $\alpha t/\lambda^2$. Comparing laws with W_t forces $\alpha = 0$. The stronger conclusion $\kappa > 0$ requires a nondegeneracy assumption.

E26. Local growth does not require self-intersection

Location: p. 25, paragraph after Theorem 14

Generating curves *may have self-contacts and need not be simple*; self-intersection is not required. A vertical slit is a simple curve generating a local-growth chain, so local growth cannot require self-intersection. The relevant noncrossing condition is that future growth stays in the closure of the current unbounded component.

E27. Boundary points and prime ends

Location: pp. 25–26, two-pointed domains and Definition 9

For an arbitrary simply connected domain, ordinary boundary points are not sufficient data for values such as $\phi(0) = a$ and $\phi(\infty) = b$: boundary limits need not exist at every point, and different prime ends can have the same Euclidean boundary point.

In a valid formulation, $U \subsetneq \mathbb{C}$ is simply connected. If U is a Jordan domain (with the spherical convention for unbounded domains), ordinary boundary points suffice. In general, a, b are distinct prime ends and ϕ is interpreted through its prime-end extension. Berestycki–Norris use an abstract conformal boundary in their definition of two-pointed domains, precisely to make these marked boundary values meaningful.

E28. Composition and time rescaling under conformal maps

Location: p. 26, comparison of two choices of conformal map

For $\phi, \psi : \mathbb{H} \xrightarrow{\sim} U$, it is

$$\psi^{-1} \circ \phi \in \text{Aut}(\mathbb{H}), \quad \psi^{-1} \circ \phi(z) = cz \quad (c > 0),$$

whereas $\phi \circ \psi^{-1}$ is an automorphism of U . Thus $\phi = \psi \circ (z \mapsto cz)$. The induced half-plane time rescaling is quadratic: by SLE scaling,

$$(\phi(K_t))_{t \geq 0} \stackrel{d}{=} (\psi(K_{c^2 t}))_{t \geq 0}.$$

The law is therefore independent of the map when curves are considered modulo increasing reparametrization. In the pushforward formula, ϕ denotes the induced map on hull families or curves, not merely the point map.

E29. Minkowski content is conformally covariant

Location: p. 26, final paragraph

Natural length is not conformally invariant. For $0 < \kappa < 8$, put $d = 1 + \kappa/8$ and let Θ be the d -dimensional Minkowski-content measure along the SLE trace. Under a conformal map ϕ , the natural clock transforms as

$$\Theta_{\phi \circ \gamma}([0, t]) = \int_0^t |\phi'(\gamma(s))|^d d\Theta_\gamma(s),$$

on curve portions where the conformal map is defined, with the usual limiting interpretation near endpoints. Even a dilation by c multiplies natural length by c^d . The appropriate property is therefore *conformal covariance*, with the parameter range as stated above. The existence and identification with natural time for $0 < \kappa < 8$ are proved by Lawler–Rezaei [R2]. The space-filling regime requires a separate area-based discussion.

Smaller formula and notation corrections

The entries below give additional corrections to formulas, notation, and bibliographic information.

Location	Correction
p. 2, last paragraph before Prop. 2	The derivative moduli of F and $F \circ (e^{i\theta} \text{id})$ agree at 0. They need not agree at the same nonzero point. The identity is $ G'(0) = F'(0) $.
p. 3, Prop. 3 proof	The normalization is $f'_2(z_0) > 0$, where $f_2(z_0) > 0$ is printed. The displayed value $f_2(z_0) = 0$ is otherwise contradictory.
p. 3, heading	The standard term is <i>hydrodynamic normalization</i> , rather than thermodynamic normalization.
p. 4, scaled Laurent series	The coefficient of r^{-1} is a_{-1}/a_1 , not a_{-1}/a_0 .
p. 6, Koebe diagram	The image is $\mathbb{C} \setminus (-\infty, -1/4]$, not $\mathbb{C} \setminus (-\infty, 1/4]$.
p. 9, Lemma 1	The analytic square root g is chosen with $g(0) = 1$, which ensures $h'(0) = 1$. In the injectivity argument, if $z_1 = -z_2$ and $h(z_1) = h(z_2)$, then both values vanish, so $z_1 = z_2 = 0$.
p. 9, coefficient comparison	The coefficient of z^4 in $(z + b_2 z^2 + b_3 z^3 + \cdots)^2$ is $2b_3 + b_2^2$, not $2b_3 + b_2$.
p. 10, quarter-theorem proof	The relation is $w \notin \frac{1}{4}\mathbb{D}$ (non-membership, rather than non-inclusion). The triangle-inequality chain ends with ≤ 4 , not ≥ 4 . The equality case gives a normalized rotation of the Koebe function, not necessarily the unrotated one.
p. 11, Cor. 1 proof	The inclusion is $\mathbb{D} \subseteq \Omega = f(\mathbb{D})$, where $\mathbb{D} \subseteq f(\Omega)$ is printed.
p. 20, Definition 8	$(K_t)_{t \geq 0}$ is an $\mathcal{L}$ -valued random variable, equivalently a hull-valued stochastic process. Individual K_t are hulls, not elements of the path space $\mathcal{L}$.
p. 22, differential estimates	The chain index is k throughout. The differentiation and integration variable is s , and the integration differential is ds , where dt is printed.
p. 25, before Thm. 14	The notation is $H_t = \mathbb{H} \setminus K_t$.
p. 27, Brownian-motion reference	The authors are Peter Mörters and Yuval Peres [R3].

Conventions and qualifications

C1. Carathéodory convergence (p. 20, Proposition 13). In E20–E22, convergence of normalized half-plane hulls means locally uniform convergence of g_K^{-1} . Identifying this with ordinary pointed kernel convergence of domains requires additional control of the normalization at infinity.

For example, let $K_n = [n, n + in]$. The domains $\mathbb{H} \setminus K_n$ converge to $\mathbb{H}$ in the ordinary pointed kernel sense, whereas

$$g_n(z) = n + \sqrt{(z - n)^2 + n^2} = (1 - \sqrt{2})n + z/\sqrt{2} + O(n^{-1})$$

for each fixed $z \in \mathbb{H}$, with the mapping-out branch. These maps do not converge to z . A common bounded region containing the hulls is a standard sufficient extra control in applying the usual kernel theorem to hydrodynamically normalized maps. The escaping-slit sequence does not converge in the topology defined through normalized inverse maps.

C2. Raw and augmented Brownian filtrations (pp. 16–17). The distinction between $\mathcal{F}^0$ and $\mathcal{F}^+$ is not itself an error. On canonical continuous-path space, raw natural sigma-algebras need not be right-continuous. The usual completed Brownian filtration is right-continuous. Any assertion of strict enlargement concerns the raw filtration, not the usual augmented filtration. In the discussion of entry into an open set, immediate entry means visits at arbitrarily close later times, not necessarily residence in the set for an entire right-neighborhood.

C3. The capacity convention (p. 12). The notes use $\text{cap}(K) = \log a_1$. With this convention, the usual potential-theoretic logarithmic capacity is $a_1 = e^{\text{cap}(K)}$. The difference is one of convention rather than a sign error. Likewise, $f_K(z) = 1/F_K(1/z)$ is holomorphic on the disc when $0 \in K$; if $0 \notin K$, it has a pole, so it is not an element of S .

C4. The claimed domain correspondence (p. 11). The normalized class S^* corresponds to proper simply connected domains *containing zero*. The set $\mathcal{A}$ defined on p. 6 includes domains not containing zero, so there is no bijection with all of $\mathcal{A}$ as written. The translation in the proof of Corollary 2 resolves this locally. The general correspondence holds for pointed domains, or after restricting $\mathcal{A}$ to domains containing zero.

References

- [R1] Nathanaël Berestycki and James Norris, *Lectures on Schramm–Loewner Evolution*, 22 August 2023. Sections 3–5; especially Theorem 4.11 and the definitions of hull topology and two-pointed domains. This is a later version of the lecture notes cited in Zhang’s bibliography. The counterexample in E22 addresses the topology as explicitly stated there.
[Author-hosted PDF](#).
- [R2] Gregory F. Lawler and Mohammad A. Rezaei, *Minkowski content and natural parameterization for the Schramm–Loewner evolution*, *Annals of Probability* **43** (2015), 1082–1120.
[arXiv:1211.4146](#); [doi:10.1214/13-AOP874](#).
- [R3] Peter Mörters and Yuval Peres, *Brownian Motion*, Cambridge University Press, 2010. Correct bibliographic attribution for the Brownian-motion reference in the notes.
[Cambridge University Press record](#).

- [R4] Antti Kemppainen, *Schramm–Loewner Evolution*, SpringerBriefs in Mathematical Physics **24**, 2017. Relevant passages are Theorem 4.2, p. 73, and Section 6.1 (pp. 102–106). Proposition 6.2 has a fixed containing-hull hypothesis absent from the formulation on p. 73.
[doi:10.1007/978-3-319-65329-7](https://doi.org/10.1007/978-3-319-65329-7).
- [R5] Fredrik Johansson Viklund, Steffen Rohde and Carto Wong, *On the continuity of SLE_κ in κ* , Probability Theory and Related Fields **159** (2014), 413–433. Lemma 2.3, particularly (2.6), proves an inverse-map perturbation estimate; the preceding paragraph also discusses L^1 perturbations. This is reference [5] in Kemppainen’s Chapter 6.
[arXiv:1206.1956](https://arxiv.org/abs/1206.1956); [doi:10.1007/s00440-013-0506-z](https://doi.org/10.1007/s00440-013-0506-z).
- [R6] Gregory F. Lawler, Oded Schramm and Wendelin Werner, *Values of Brownian intersection exponents I: Half-plane exponents*, Acta Mathematica **187** (2001), 237–273. Theorem 2.6 in Section 2.3 gives and proves the chordal correspondence. It cites Ch. Pommerenke, *On the Loewner differential equation*, Michigan Mathematical Journal **13** (1966), 435–443, for the disk analogue.
[arXiv:math/9911084](https://arxiv.org/abs/math/9911084); [doi:10.1007/BF02392618](https://doi.org/10.1007/BF02392618).