Loop-Erased Random Walks

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This project aims to study the loop-erased random walk (LERW) with an emphasis towards its definition on $\mathbb{Z}^2$, prepare for further reading on its scaling limit to SLE_2, and explore its relationship with uniform spanning trees (UST).

We will outline the combinatorial properties of loop-erasure and define the LERW for killed Markov chains on finite graphs, as well as for transient and some recurrent chains on infinite graphs. For finite and transient chains, we will explore the three forms of the LERW: the loop-erasure of the chain killed at the boundary/the infinite chain, a non-Markovian process defined on the original chain, and the Laplacian random walk. For the recurrent case, we make sense of the latter two forms for chains that satisfy a specific property. Then, we will define the UST on finite graphs and introduce Wilson's algorithm, which connects the UST to the LERW. We will move to wired boundary conditions on finite sets and consider the wired spanning tree (WSF), which will allow us to define the uniform spanning forest (USF) on $\mathbb{Z}^d$, using an infinite version of Wilson's algorithm. Finally, we will return to the issue of defining the LERW on more general recurrent chains, where we further explore its close connection with the UST.

The loop-erased random walk (LERW) is a model of non-self-intersecting paths on graphs. It was initially introduced by Lawler in [Law80] to understand the self-avoiding walk, a different measure on the set of non-self-intersecting paths. Although it turned out that these models are not very similar, the LERW became an interesting model to study in its own right, in part due to its rich connections to the Schramm-Loewner evolution, discovered by Schramm in [Sch00], and uniform spanning forests as in [Pem91]. The latter connection will be explored in this report as well. Except for the last section, we use [Law17] and [Law22] as main references.

Markov Chains on Graphs

We will start in a general setting. Let A and ∂A denote finite vertex sets, where ∂A should be interpreted as the external boundary of A. We write $\overline{A} = A \cup \partial A$. We may also use $A, \partial A, \overline{A}$ to refer to the complete graphs generated by these vertex sets. Let S denote an irreducible Markov chain on $\overline{A}$ with transition probabilities $(p(x, y))_{x,y\in\overline{A}}$, killed upon exiting A. We will use P to denote its transition matrix and $\mathbf{P}^x$ for the associated measure when starting at $x \in A$. Let $\tau_A = \inf\{n \geq 0\,;\, S_n \notin A\}$ denote the exit time of A and $\tilde{H}_B = \inf\{n \geq 1\,;\, S_n \in B\}$ denote the hitting time of $B \subseteq A$.

We will consider sets of paths quite frequently. For $x, y \in \overline{A}$, we write $\mathcal{K}_A(x, y)$ for the set containing paths of the following form: $\omega = [\vec{e}_1, \ldots, \vec{e}_n]$, where $\vec{e}_1$ points away from x, $\vec{e}_n$ points to y, and $\vec{e}_j$ has both vertices in A for any $2 \leq j \leq n-1$. We also define

$$\mathcal{K}_A = \bigcup_{x\in A}\bigcup_{y\in A}\mathcal{K}_A(x, y) \quad \text{and} \quad \overline{\mathcal{K}_A} = \bigcup_{x\in\overline{A}}\bigcup_{y\in\overline{A}}\mathcal{K}_A(x, y). \tag{1}$$

We will also encounter self-avoiding walks (SAW), which are paths that do not visit the same vertex twice. For sets of SAWs, we use similar notations to sets of usual paths, replacing the letter $\mathcal{K}$ with $\mathcal{W}$ instead, e.g. $\mathcal{W}_A(x, y)$.

Usually, we define Markov chains via their transition probabilities. For Markov chains on graphs, a different perspective may be easier to work with. Given a directed edge $\vec{e} = (x, y)$, we define a weight on $\vec{e}$, given by $p(\vec{e}) = p(x, y)$. This induces a measure p on sets of paths, e.g. $\mathcal{K}_A(x, y)$.

We will also study Markov chains on countably infinite graphs. Throughout this report, we will use X to denote a countably infinite vertex set. We say that an increasing sequence $(A_n)_{n\in\mathbb{N}}$ of finite sets is an exhaustion of X if their union is X. The standard example of these graphs is $\mathbb{Z}^d$. We recall that the simple random walk is recurrent for $d = 1, 2$ and transient for $d \geq 3$.

Potential Theory

The LERW is deeply related to potential theory on graphs. Analogous to the continuum, we have the main objects: the Laplacian, Green's function, and the Poisson kernel.

Definition 1. Given a Markov chain S with the transition matrix P, its Laplacian is the operator $\Delta = P - I$. If f is a function on $\overline{A}$, we say that f is harmonic if $\Delta f = 0$ on A.

The Green's function $G_A(x, y)$ killed upon exiting A is the expected number of visits of the chain started at x to y before exiting A, where $x, y \in A$. It can be interpreted as an operator G_A.

The Poisson kernel is given by $H_A(x, z) = \mathbf{P}^x \left\{S_{\tau_A} = z\right\}$, where $x \in A$ and $z \in \partial A$.

The Laplacian and Green's function satisfy the relationship $\Delta = -G_A^{-1}$. Now, we will look at the first application of taking p as a measure on sets of paths.

Proposition 1 ([Law17, p.7, 9])**.** For any $x, y \in A$ and $z \in \partial A$,

$$G_A(x, y) = p[\mathcal{K}_A(x, y)] \quad \text{and} \quad H_A(x, z) = p[\mathcal{K}_A(x, z)]. \tag{2}$$

We will also encounter functions which are harmonic on a finite subset which satisfy certain boundary conditions. They are analogous to solutions to the Dirichlet problem in PDE theory and are referred to by the same name. The following theorem gives an explicit solution to this problem in terms of the Poisson kernel.

Theorem 1 ([Law22, Proposition 1.10])**.** Suppose P is an irreducible transition matrix on $\overline{A}$, and let $F : \partial A \to \mathbb{R}$ be a bounded function. Then there exists a unique bounded function f on $\overline{A}$ that is harmonic in A and agrees with F on ∂A. It is given by

$$f(x) = \mathbf{E}^x \left[F(S_{\tau_A})\right] = \sum_{z \in \partial A} H_A(x, z) F(z). \tag{3}$$

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Deterministic Paths

Definition 2. We define a deterministic procedure called (chronological) loop-erasure

$$\mathsf{LE} : \omega \in \mathcal{K}_A(x, y) \mapsto \eta \in \mathcal{W}_A(x, y). \tag{4}$$

It is given by the following:

1. Let $\omega = [\omega_0, \ldots, \omega_n] \in \mathcal{K}_A(x, y)$, $j_0 = \sup\{k \,;\, \omega_k = \omega_0\}$, and $\eta_0 = \omega_0 = \omega_{j_0}$.

2. If $j_i < n$, set $j_{i+1} = \max\left\{k\,;\,\omega_k = \omega_{j_i+1}\right\}$, and $\eta_{j+1} = \omega_{j_i+1} (= \omega_{j_{i+1}})$. Recursively apply this procedure until $j_m = n$.

3. Set $\mathsf{LE}(\omega) = \eta = [\eta_0, \ldots, \eta_m]$.

We study the properties of this procedure. Note that as a function, loop-erasure is not injective. Let us examine some properties of $\mathsf{LE}^{-1}(\eta)$, the set of paths with loop-erasure η.

Definition 3. Given a weight p on A, the loop-erased measure $\hat{p}$ is a measure on $\overline{\mathcal{W}}_A$, given by

$$\hat{p}(\eta) = p(\mathsf{LE}^{-1}(\eta)) = \sum_{\omega \in \overline{\mathcal{K}}_A, \mathsf{LE}(\omega)=\eta} p(\omega) \tag{5}$$

The measure $\hat{p}$ has a more explicit form in terms of Green's functions and the usual measure p.

Proposition 2 ([Law17, Proposition 3.1])**.** If $\eta = [x_0, \ldots, x_m] \in \mathcal{W}_A$, then

$$\hat{p}(\eta) = p(\eta) \prod_{j=0}^{m} G_{A_j}(x_j, x_j), \quad A_j = A \setminus \left\{x_0, \ldots, x_{j-1}\right\}. \tag{6}$$

It is useful to think of the loop-erasure η as a 'backbone', such that when it is combined with the loops erased, we recover the original path. In this sense, the quantity $\prod G_{A_j}(x_j, x_j)$ is the measure of all possible combinations of loops which one can put on the SAW η. This quantity is so important that we now generalise it and give it a notation.

Definition 4. Let $B = \{x_1, \ldots, x_n\} \subseteq A$. Then

$$F_B(A) = \prod_{j=1}^{n} G_{A_j}(x_j, x_j), \quad A_j = A \setminus \left\{x_0, \ldots, x_{j-1}\right\}. \tag{7}$$

If $B \not\subset A$, we define $F_B(A) = F_{B\cap A}(A)$. We also write $F(A) = F_A(A)$.

It may be surprising to learn that the quantity $F_B(A)$ is independent of the enumeration of B. In particular, we have the identity $F(A) = \det G_A = 1/\det(I - P)$.

Finite State Spaces

Definition 5. Let A be finite, ∂A nonempty, and S an irreducible Markov chain on $\overline{A}$. Let $x \in A$. The LERW from x to ∂A to is a probability measure $\hat{p}$ on $\mathcal{W}_A(x, \partial A)$, obtained by starting the Markov chain at x, killing the chain upon leaving A, and erasing loops chronologically. More explicitly,

$$\hat{p}(\eta) = \mathbf{P}\left\{\mathsf{LE}([S_0, \ldots, S_{\tau_A}]) = \eta\right\}. \tag{8}$$

This coincides with the loop-erased measure, giving the probability $\hat{p}(\eta) = p(\eta)F_\eta(A)$.

Alternatively, we can define the non-Markovian process $\hat{S}$ by $\hat{S}_n = S_{T_n}$, stopped when reaching the boundary, for random times T_n defined by

$$T_0 = 0, \quad T_n = \sup\left\{k \le \tau_A \,;\, S_k = S_{T_{n-1}+1}\right\}. \tag{9}$$

This is the LERW on the Markov chain S, from the starting point of S to ∂A. The trace of $\hat{S}$ is indeed loop-erasure of that of S. We are interested in the probability that $\hat{S}$ starts with some $\eta \in \mathcal{W}_A$.

Proposition 3 ([Law17, Proposition 4.4])**.** Suppose $\eta = [x_0, \ldots, x_n]$, for some $x_0, x_n \in A$. Then

$$\mathbf{P}^x\left\{[\hat{S}_0, \ldots, \hat{S}_n] = \eta\right\} = p(\eta)F_\eta(A)\mathrm{Es}_\eta(\eta_k), \tag{10}$$

where $\mathrm{Es}_B(x) = \mathbf{P}^x\{S_n \notin B \text{ for all } 1 \le n \le \tau_A\}$, is the escape probability, for $B \subseteq A, x \in \overline{A}$.

It is worthwhile to delve deeper into the probability $\mathrm{Es}_B(x)$. We also define

$$\phi_B(x) = \mathbf{P}^x\{S_n \notin B \text{ for all } 0 \le n \le \tau_A\} \quad \forall B \subseteq A, x \in \overline{A}. \tag{11}$$

This is harmonic on $A \setminus B$ and satisfies the boundary data given by $\mathbf{1}_{\partial A}$ on $\partial(A \setminus B) = \partial A \cup B$. It is the unique solution to this Dirichlet problem. The escape probability can be retrieved from this by the following:

$$\mathrm{Es}_B(x) = \begin{cases} \phi_B(x) & x \notin B, \\ \Delta\,\phi_B(x) & x \in B. \end{cases} \tag{12}$$

The function ϕ_B is an object solely determined by the potential theory on A given the weight p. From the above proposition, we can obtain the 'transition probabilities' of $\hat{S}$ as a non-Markovian process.

$$\mathbf{P}^{x_0}\left\{\hat{S}_{n+1} = z \mid [\hat{S}_0, \ldots, \hat{S}_n] = \eta\right\} = \frac{p(x_n, z)\phi_\eta(z)}{\sum_{w \in \mathcal{X}} p(x_n, w)\phi_\eta(w)}. \tag{13}$$

This is the 'Laplacian random walk' interpretation of the LERW, which is another characterisation. It is named as such as each step is taken according to ϕ_η, which is the solution to Laplace's equation $\Delta\,\phi_\eta = 0$.

Transient Chains on Infinite State Spaces

Let S denote an irreducible transient Markov chain on a countably infinite state space X. It is easiest to define the LERW by the process $\hat{S}$ given by $\hat{S}_n = S_{T_n}$, for random times T_n defined by

$$T_0 = 0, \quad T_n = \sup\left\{k \in \mathbb{N}_0 \,;\, S_k = S_{T_{n-1}+1}\right\}. \tag{14}$$

By transience, the times T_n are almost surely finite.

The LERW on transient chains also exhibits analogous properties to that in finite state spaces. Although it is interesting in its own right, we will only use it as a tool for constructing uniform spanning forests in this report.

Recurrent Chains

Let S denote an irreducible recurrent Markov chain on a countably infinite state space X. We cannot define LERW on X simply by erasing loops from the infinite path because S visits any point infinitely many times. We need to find an alternative definition. As we will see, it is only possible to define the LERW for some recurrent chains. We now develop a certain property which is satisfied by some Markov chains for which the LERW can be defined.

Let $x_0 \in X$ and $(A_n)_{n\in\mathbb{N}}$ an exhaustion of X with $x_0 \in A_0$. Also let $\eta = [x_0, \ldots, x_m]$ denote a (finite) SAW in X starting at x_0. From the previous section, we would like to define

$$\hat{p}(\eta) = \lim_{n\to\infty} p(\eta) F_\eta(A_n) \mathrm{Es}_{A_n}(x_m). \tag{15}$$

We investigate the existence of this limit. Recall the following:

$$F_\eta(A_n) = G_{A_n}(x_0, x_0) \prod_{k=1}^{m} G_{A_n\setminus\{x_0,\ldots,x_{k-1}\}}(x_k, x_k) = G_{A_n}(x_0, x_0) F_\eta(A_n \setminus \{x_0\}). \tag{16}$$

By recurrence, $G_{A_n}(x_0, x_0)$ tends to infinity. However, for any $y \in X$, $G_{A_n\setminus\{y\}}(x_0, x_0)$ is finite.

Proposition 4. Let S denote a Markov chain on a countably infinite state space X. Then for any $x, y \in X$, $G_{X\setminus\{y\}}(x, x)$ is finite.

In particular, $F_\eta(A_n \setminus \{x_0\})$ is finite. Hence, as n tends to infinity, $G_{A_n}(x_0, x_0)$ tends to infinity and $\mathrm{Es}_{A_n}(x_m)$ tends to zero. Since $F_\eta(A_n) = G_{A_n}(x_0, x_0) F_\eta(A_n \setminus \{x_0\}) \sim G_{A_n}(x_0, x_0) F_\eta(\mathcal{X} \setminus \{x_0\})$, we have

$$\hat{p}(\eta) = p(\eta) F_\eta(\mathcal{X} \setminus \{x_0\}) \left[\lim_{n\to+\infty} G_{A_n}(x_0, x_0) \mathrm{Es}_{\eta, A_n}(x_m)\right], \tag{17}$$

if the limit exists.

We are ready to introduce **Property A**, which states that this limit always exists.

Definition 6. Let V be a finite subset of X and $y \in V$. Let A_n denote an exhaustion of X and ϕ_n the function that is harmonic on $A_n \setminus V$, vanishes on V, and takes value 1 on $X \setminus (A_n \cap V)$. We define the function

$$\phi_{V,y}(x) = \lim_{n\to\infty} G_{A_n}(y, y)\phi_n(x). \tag{18}$$

A Markov chain is said to satisfy **Property A** if $\phi_{V,y}(x)$ is well-defined, i.e. invariant under change of A_n, vanishes on V, and is harmonic on $X \setminus V$.

From the previous discussion, we also obtain the appropriate definition of the LERW.

Definition 7. For a recurrent irreducible Markov chain S that satisfies **Property A**, we define the infinite LERW starting at x_0 by

$$\mathbf{P}\left\{[\hat{S}_0, \ldots, \hat{S}_n] = \eta\right\} = p(\eta)F_\eta(X \setminus \{x_0\})[\Delta\, \phi_{\eta,x_0}(x_n)], \tag{19}$$

where $\eta = [x_0, \ldots, x_n]$.

It is possible to show that the simple random walk on $\mathbb{Z}$ does not satisfy **Property A**, while that on $\mathbb{Z}^2$ does. This is the setting for Schramm's celebrated paper [Sch00] on the scaling limits of the LERW and the UST in the plane.

We could have defined the LERW on recurrent chains as the limits of the laws of LERWs over exhaustions of X. From the discussion, we see that **Property A** is guarantees the existence of this limit through the convergence of $G_{A_n}(y, y)\phi_n(x)$. This approach is very explicit and allows us to work quantitatively. More fundamentally, the existence of this limit can be characterised by a topological feature of the associated uniform spanning tree, which we will expand on at the end of this report.

Uniform Spanning Forests

We first consider uniform spanning forests on finite graphs. We enumerate $A = \{x_0, x_1, \ldots, x_n\}$.

A spanning tree T of A (where A is seen as a complete graph) is a collection of edges such that the graph with vertex set A and edges T is connected and contains no cycles.

Given a spanning tree T and a vertex $y \in A$, there exists a unique SAW $\eta \in \mathcal{W}_A(y, x_0)$ with edges in T. With this fact, it is easy to see that this defines an orientation of the tree by orienting each SAW to towards the root x_0. We define the weight of T with respect to ∂A by

$$p(T; x_0) = \prod_{e \in T} p(\vec{e}), \tag{20}$$

where $\vec{e}$ is a directed edge with direction given by the orientation of the tree.

Definition 8. The uniform spanning tree (UST) on A is a measure μ_A on the set $\mathfrak{T}$ of spanning trees of A given by

$$\mu_A(T) = \frac{p(T; x_0)}{\sum_{T \in \mathfrak{T}} p(T; x_0)}. \tag{21}$$

Although it is named as such, we see from the definition that the UST is not necessarily the 'uniform distribution' on the set of spanning trees, except when p comes from the simple random walk. To see this, recall that for a tree T, the sizes of the vertex set $V = A$ and edge set $E = T$ satisfy the relation $|V| = |E| + 1$. In this case, this shows $p(T; x_0) = p^{|A|-1}$, which is constant.

Let us move to a slightly different setting. We consider the graph $\overline{A} = A \cup \partial A$ with *wired boundary conditions* by identifying all vertices in ∂A to become one point, which we denote by ∂A. The edges that originally have endpoints at some $z \in \partial A$ now end in the vertex ∂A. That is, we are now working on the graph $\overline{A} = A \cup \{\partial A\}$ and our trees are rooted at ∂A with weights $p(T; \partial A)$. The UST on $\overline{A}$ is called the wired spanning forest (WSF) of A, although it is always a tree in $\overline{A}$.

The WSF of A restricts to A, by taking only edges with both endpoints in A. This gives us the uniform spanning forest USF of the finite set A, which is truly a forest as it is disconnected in general. It is important to note that the USF in this sense is neither uniform in the sense of a uniform distribution, nor in the sense of being proportional to its weight as in the UST.

Wilson's Algorithm

The UST and the LERW are closely related through Wilson's algorithm, which is an algorithm to generate spanning trees given instances of loop-erased random walks.

Definition 9. Given $A, \mathbf{P}, x_0 \in A$, we describe Wilson's algorithm to generate a spanning tree of A.

1. Run a LERW in A from x_1 to x_0. Let $\mathcal{T}_1$ denote the set of edges traversed by the LERW and $A_1 = A \setminus \mathcal{T}_1$.

2. Recursively, if A_k is empty, then we have a spanning tree $\mathcal{T} \stackrel{\text{def}}{=} \mathcal{T}_k$. Stop.

3. Otherwise, let j denote the smallest index with $x_j \notin A_k$. Run a LERW in A_k from x_j to $\mathcal{T}_k$. Add the edges traversed by the LERW to the tree $\mathcal{T}_k$ to form a new tree $\mathcal{T}_{k+1}$ and let $A_{k+1} = A_k \setminus \mathcal{T}_{k+1}$, i.e. the vertices that are not yet connected to the tree.

We see that this algorithm terminates in at most n steps, since at each step, at least one vertex x_j is added to A_k. This algorithm generates spanning trees in the same distribution as the UST.

Theorem 2 ([Law17, Proposition 4.7])**.** Given $A, \mathbf{P}, x_0 \in A$, the probability that a particular spanning tree $\mathcal{T}$ is generated in Wilson's algorithm is $p(\mathcal{T}; x_0)F(A)$.

When A has wired boundary conditions, Wilson's algorithm generates the WSF on A, a UST rooted at $\{\partial A\}$. For an exhaustion $(A_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}^d$, we can consider the limit of the WSFs as n tends to infinity. The boundaries ∂A_n are pushed to infinity. Intuitively, this suggests that the limit of the WSFs is a WSF rooted at infinity. While it is possible to make this limit precise, we will make use of the LERW machinery which we have developed to generalise the WSF to the infinite graph $\mathbb{Z}^d$ with the infinite Wilson's algorithm.

Definition 10 (infinite Wilson's algorithm)**.** For $d \geq 2$, we define the uniform spanning forest (USF) $\mathcal{T} = \cup_{k=1}^{\infty} \tilde{\mathcal{T}}_k$, where $\tilde{\mathcal{T}}_k$ are given by the following:

1. Run the LERW $\hat{S}^1$ generated by the simple random walk S^1 from x_1 to infinity (recall that we have a special LERW for $d = 2$). We include all edges and vertices of $\hat{S}^1[0, \infty)$ in the initial tree $\tilde{\mathcal{T}}_1$.

2. Recursively, if $x_j \in \tilde{\mathcal{T}}_{j-1}$, we set $\tilde{\mathcal{T}}_j = \tilde{\mathcal{T}}_{j-1}$. If x_j is not already included in $\tilde{\mathcal{T}}_{j-1}$, we run the simple random walk S^j until it hits $\tilde{\mathcal{T}}_{j-1}$, or until infinity if it does not. Then, we add the edges and vertices of either the loop-erasure $\mathsf{LE}(S^j[0, \tilde{H}_{\tilde{\mathcal{T}}_{j-1}}])$, or the infinite SAW $S^j[0, \infty)$ to the forest $\tilde{\mathcal{T}}_{j-1}$ to form $\tilde{\mathcal{T}}_j$.

Is the USF truly a forest, or is it a tree? To examine the connectivity of the USF constructed by the infinite Wilson's algorithm, we need to consider the probability that a loop-erased random walk and an independent simple random walk do not intersect.

Proposition 5 ([Pem91, Lemma 3.1])**.** If S^1, S^2 are independent simple random walks starting at $x, y \in \mathbb{Z}^d$ respectively, then

$$\mathbf{P}\left\{\hat{S}^1[0,\infty) \cap S^2[1,\infty) = \emptyset\right\} \begin{cases} = 0 & d \leq 4, \\ > 0 & d \geq 5. \end{cases} \tag{22}$$

With this result, it is easy to show the following by considering Wilson's algorithm.

Theorem 3 ([Law22, Proposition 5.31])**.** For $d = 2, 3, 4$, the USF $\mathcal{T}$ is almost surely connected, i.e. it is almost surely a tree. For $d \geq 5$, $\mathcal{T}$ almost surely has infinitely many components.

Reprise: LERW on Recurrent Chains

In this section, we reference [BLPS01].

We have defined the USF in $\mathbb{Z}^d$ via the infinite Wilson's algorithm, but this is not very satisfactory as it relies on LERWs, which do not always exist for recurrent graphs. As we have mentioned, it is possible to define the USF in general graphs by taking limits of WSFs over exhaustions. It can also be shown that if the LERW exists, then the random spanning forest generated by Wilson's algorithm is indeed the USF.

For recurrent chains, it is obvious from Wilson's algorithm that the USF is the UST. We now define the topological notion of ends for spanning trees. An infinite SAW in a tree is called a ray. Two rays are equivalent if they coincide on infinitely many vertices. An end is an equivalence class of rays. Observe that once two equivalent rays coincide, they must follow the same path. If not, they will separate and coincide at another point, creating a cycle, which cannot happen in a tree. In the infinite Wilson's algorithm on $\mathbb{Z}^d$, we saw that every spanning tree generated has an end generated by the initial LERW. In fact, if the UST has only one end, then we can recover the LERW from it.

Theorem 4 ([BLPS01, Proposition 14.1])**.** Let X denote an infinite graph, $(A_n)_{n\in\mathbb{N}}$ an exhaustion of X, and S^o a recurrent Markov chain on X started at some point $o \in G$. Let $\eta_n = [\hat{S}^o_0, \ldots, \hat{S}^o_{\tau_{A_n}}]$. If the UST $\mathcal{T}_X$ has only one end a.s., then the law of the random path η_n converges weakly to the law of the unique ray from o in $\mathcal{T}_X$.

In particular, this is true for any recurrent chain on a proper planar graph which has only finitely many sides to each face, where a planar graph is proper if every bounded set in the plane contains only finitely many edges and vertices.

References

[BLPS01] Benjamini I, Lyons R, Peres Y, Schramm O. Uniform spanning forests. The Annals of Probability. 2001;29(1):1-65. doi:10.1214/aop/1008956321

[Law80] Lawler GF. A self-avoiding random walk. Duke Mathematical Journal. 1980;47(3):655-693. doi:10.1215/S0012-7094-80-04741-9

[Law17] Lawler GF. Topics in loop measures and the loop-erased walk. Published online 2017. https://arxiv.org/abs/1709.07531

[Law22] Lawler GF. Random Explorations. American Mathematical Society; 2022.

[Pem91] Pemantle R. Choosing a Spanning Tree for the Integer Lattice Uniformly. The Annals of probability. 1991;19(4):1559-1574. doi:10.1214/aop/1176990223

[Sch00] Schramm O. Scaling limits of loop-erased random walks and uniform spanning trees. Israel Journal of Mathematics. 2000;118(1):221-288. doi:10.1007/BF02803524