The Loop-Erased Random Walk

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This project aims to study the loop-erased random walk on general graphs, with an emphasis on $\mathbb{Z}^2$, as well as related topics such as uniform spanning trees and loop soups. A key objective is to understand the scaling limit of loop-erased random walks on $\mathbb{Z}^2$ and its relation to the Schramm-Loewner evolution. In this report, we outline the main constructions, present some key ideas, and highlight interesting results which emerge from studying the loop-erased random walk.

The loop-erased random walk (LERW) was first introduced by Lawler[1] to understand the self-avoiding walk. Since then, it has been a major subject of interest, particularly due to its connections with other objects, especially the uniform spanning tree[2] and the Schramm-Loewner evolution.[3]

Summary

Our main reference is *Topics in loop measures and the loop-erased walk* (2017)[4] by Greg Lawler, with a supplementary reference *Random Explorations* (2022)[5] by the same author.

Let A and ∂A be finite vertex sets, where ∂A is understood to be the external boundary of A. Let $\overline{A} = A \cup \partial A$. We also use $A, \partial A, \overline{A}$ to denote the complete graphs generated by these vertex sets. We write $\mathcal{K}_A(x, y)$ for the set of paths from x to y in A, and $\overline{\mathcal{K}}_A$ for paths which can start and end anywhere in $\overline{A}$, with all other vertices in A . Let S denote an irreducible Markov chain on $\overline{A}$ with transition probabilities $p(x, y)$. We require $p(z, z) = 1$ for any $z \in \partial A$, effectively ending the path as it leaves A.

The transition probabilities in $\overline{A}$ induce a measure on paths in $\overline{A}$, given by

$$p(\omega) = \prod_{i=0}^{n-1} p(x_i, x_{i+1}), \quad \text{where} \quad \omega = [x_0, \dots, x_n]. \tag{1}$$

This measure is equivalent to the Markov chain S. In particular, $\mathbf{P}^{x_0}\{[S_0, \dots, S_n] = \omega\} = p(\omega)$. The classical quantities in the theory of random walks: the Green's function G_A, and the Poisson kernel H_A, can be rewritten as measures of sets of paths:

$$G_A(x, y) = p[\mathcal{K}_A(x, y)], \quad \text{for} \quad x, y \in A, \tag{2}$$

$$H_A(x, z) = p[\mathcal{K}_A(x, z)], \quad \text{for} \quad x \in A, z \in \partial A. \tag{3}$$

In the following, we will set up an analogous correspondence between the loop-erased random walk and the loop-erased measure. We first look at erasing loops on deterministic paths.

Definition 1 *A path $\omega = [\omega_0, \dots, \omega_n]$ is called a self-avoiding walk (SAW) if all its vertices are distinct. We will write $\eta = [\eta_0, \dots, \eta_m]$ for SAWs, and $\mathcal{W}_A(x, y) = \{\omega \in \mathcal{K}_A(x, y) : \omega \text{ is a SAW}\}$.*

Definition 2 *We define a deterministic procedure called (chronological) loop erasure*

$$LE : \omega \in \mathcal{K}_A(x, y) \mapsto \eta \in \mathcal{W}_A(x, y). \tag{4}$$

It is given by the following:

1. *Let $\omega = [\omega_0, \dots, \omega_n] \in \mathcal{K}_A(x, y)$, $j_0 = \sup\{k : \omega_k = \omega_0\}$, and $\eta_0 = \omega_0 = \omega_{j_0}$.*
2. *If $j_i < n$, set $j_{i+1} = \max\left\{k : \omega_k = \omega_{j_i+1}\right\}$, and $\eta_{j+1} = \omega_{j_i+1} (= \omega_{j_{i+1}})$. Recursively apply this procedure until $j_m = n$.*
3. *Set $LE(\omega) = \eta = [\eta_0, \dots, \eta_m]$.*

As a function, LE is surjective, but not injective. How do we understand the preimage $LE^{-1}(\omega)$? This is a recurring component in proofs of results regarding loop-erased (random) walks. We now demonstrate one of them.

With a measure p induced by a Markov chain, we can define the loop-erased measure $\hat{p}$ on $\overline{\mathcal{W}}_A$, given by

$$\hat{p}(\eta) = \sum_{\omega \in \overline{\mathcal{K}}_A : LE(\omega) = \eta} p(\omega). \tag{5}$$

The loop-erased measure $\hat{p}$ has an explicit form in terms of Green's functions. Given some SAW $\eta = [\eta_0, \dots, \eta_m]$ and some path $\omega \in LE^{-1}(\eta)$, we have a unique decomposition

$$\omega = \ell_0 \oplus [\eta_0, \eta_1] \oplus \ell_1 \oplus [\eta_1, \eta_2] \oplus \cdots \oplus [\eta_{m-1}, \eta_m] \oplus \ell_m, \tag{6}$$

where $\ell_j \in \mathcal{K}_{A_j}(\eta_j, \eta_j)$ and $A_j = A \setminus \{\eta_0, \dots, \eta_{j-1}\}$. From this decomposition, we obtain

$$p(\omega) = p(\eta) p(\ell_0) \dots p(\ell_m). \tag{7}$$

Summing over all possible choices of the loops $\ell_0, \dots \ell_m$, we arrive at the desired expression.

$$\hat{p}(\eta) = p(\eta) \prod_{j=0}^{m} G_{A_j}(\eta_j, \eta_j). \tag{8}$$

It is perhaps surprising to learn that the quantity $\prod G_{A_j}(\eta_j, \eta_j)$ does not depend on the order of the elements $\{\eta_0, \dots, \eta_m\}$. From this observation, we can unambiguously define the quantity

$$F_B(A) = \prod_{j=1}^{n} G_{A_j}(x_j, x_j), \tag{9}$$

where B is some subset of A with any enumeration $B = \{x_0, \dots x_n\}$, $A_j = A \setminus \{x_0, \dots, x_{j-1}\}$. This has a more fundamental description that sheds light on the independence of enumeration:

$$F_A(A) = \det G_A = \frac{-1}{\det \Delta_A}. \tag{10}$$

We are now well equipped to understand the loop-erased random walk on a finite graph $\overline{A}$. The loop-erased random walk $\hat{S}$ on the Markov chain S can be defined by starting S at some $x_0 \in A$ until it leaves A and then erasing loops. More explicitly, we write $\hat{S} = LE([S_0, \dots, S_\tau])$, where τ is the time of first exit from A. This is a non-Markovian process on $\overline{A}$, which can also be realised as a probability measure on $\mathcal{W}_A(x_0, \partial A)$:

$$\mathbf{P}^{x_0}\{\hat{S} = \eta\} = p(\eta) F_\eta(A) = \hat{p}(\eta) \quad \text{for} \quad \eta \in \mathcal{W}_A(x, z), z \in \partial A. \tag{11}$$

Thus, we have characterised the loop-erased random walk by the loop-erased measure.

The loop-erased random walk is defined similarly for irreducible transient Markov chains on countable graphs. It can also be defined for a special class of recurrent chains, with the most prominent example being the symmetric simple random walk on $\mathbb{Z}^2$. We will continue to study these processes with a view towards understanding the scaling limit of loop-erased random walks and uniform spanning trees in dimension 2.

References

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