Let $V = \mathbb{F}_q^n$, $a_1, \dots, a_m$ nonnegative integers s.t. 2026-04-01

$a_1 + \cdots + a_m = n$.

Define $FL(a_1, \dots, a_m)$ ~~$= \{ 0 \subseteq V_1 \subseteq \cdots \subseteq V_m \subseteq V \}$~~

$= \{ (V_1, \dots, V_m) : V_i \subseteq V,\ V_i \subseteq V_{i+1}, \dim(V_i / V_{i-1}) = a_i \}$

This is the flag space of type $(a_1, \dots, a_m)$

$[n]_q := 1 + \cdots + q^{n-1}$

$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$

$$\begin{bmatrix} n \\ a_1, \dots, a_m \end{bmatrix}_q := \frac{[n]_q!}{[a_1]_q! \cdots [a_m]_q!}$$

Let $\mathcal{M}^n := \{ n \times n$ matrices over $\mathbb{F}_q$ of rank $n \}$.

Consider a map $\pi : \mathcal{M}^n \longrightarrow FL(a_1, \dots, a_m)$

$$M = \left(\begin{array}{c} M_1 \\ \hline M_2 \\ \hline \vdots \\ \hline M_m \end{array} \right) \begin{array}{l} \updownarrow a_1 \\ \updownarrow a_2 \\ \\ \updownarrow a_m \end{array} \longmapsto \pi(M), \text{ where}$$

$\pi(M) = 0 \subseteq V_1 \subseteq \cdots \subseteq V_m \subseteq V$, where $V_i = \mathrm{Row}\begin{pmatrix} M_1 \\ \vdots \\ M_i \end{pmatrix}$.

$\#(\mathcal{M}^n) = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$.

$$= q^{\frac{n(n-1)}{2}} (q^n - 1) \cdots (q - 1) = q^{\frac{n(n-1)}{2}} (q-1)^n [n]_q!$$

It is ~~clea~~ clear that π is surjective.

Given a flag $F = 0 \subsetneq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_m \subseteq V \in Fl(a_1, \dots, a_m)$,
we want to find $\#|\pi^{-1}(F)|$.

$$\#(\pi^{-1}(F)) = \underbrace{(q^{a_1}-1)(q^{a_1}-q)\cdots(q^{a_1}-q^{a_1-1})}_{a_1}$$ select a_1 vectors from V_1. $\dim V_1 = a_1$

$$\cdot \underbrace{(q^{a_1+a_2}-q^{a_1})(q^{a_1+a_2}-q^{a_1+1})\cdots(q^{a_1+a_2}-q^{a_1+a_2-1})}_{a_2}$$ select a_2 vectors from ~~$V_1 \oplus$~~ V_2 _ $\dim V_2 = a_1+a_2$

$$\cdot \ \cdots$$

$$\cdot \underbrace{(q^{n}-q^{a_1+\cdots+a_{m-1}})\cdots(q^{n}-q^{\overbrace{a_1+\cdots+a_m-1}^{=n-1}})}_{a_m}$$

$$= q^e (q^{a_1}-1)(q^{a_1-1}-1)\cdots(q-1)\cdot$$
$$(q^{a_2}-1)(q^{a_2-1}-1)\cdots(q-1)\cdot$$
$$\cdots\cdot$$
$$(q^{a_m}-1)(q^{a_m-1}-1)\cdots(q-1),$$ where

$$e = (1+2+\cdots+\overline{a_1-1}) + (a_1 + a_1+1 + \cdots + \overline{a_1+a_2-1}) + \cdots$$
$$+ \left[(a_1+\cdots+a_{m-1}) + (a_1+\cdots+a_{m-1}+1) + \cdots + n-1\right]$$
$$= \frac{n(n-1)}{2}.$$

$$= q^e (q-1)^{a_1}(1+q+\cdots+q^{a_1-1})(1+q+\cdots+q^{a_1-2})\cdots(1+q)$$

$$(q-1)^{a_2}(1+q+\cdots+q^{a_2-1})\cdots(1+q)$$

$$\vdots$$

$$(q-1)^{a_m}(1+q+\cdots+q^{a_m-1})\cdots(1+q)$$

$$= q^{\frac{n(n-1)}{2}}(q-1)^n\prod_{i=1}^{m}[a_i]_q!$$

$$\frac{\#(M^n)}{\#(\pi^{-1}(F))} = \frac{[n]_q!}{\prod_{i=1}^{m}[a_i]_q!} = \begin{bmatrix} n \\ a_1,\ldots,a_m \end{bmatrix}_q$$