Lecture 1: Local Central Limit Theorem.

<u>Thm</u> X centred, $Var[X] = \sigma^2$, $S_n = \sum_{i=1}^{n} X_i$, $\mathbb{Z}^d$-valued, aperiodic. $\mathbb{E}[|X|^3] = \rho < +\infty$

$$\boxed{n^{\frac{d}{2}} \sup_{x \in \mathbb{R}^d} \left| \mathbb{P}[S_n = \lfloor x \rfloor] - \frac{1}{(2\pi\sigma^2 n)^{\frac{d}{2}}} \exp\left\{ -\frac{|x|^2}{2n\sigma^2} \right\} \right| \le O\left(n^{-\frac{d}{2}+\varepsilon}\right) \qquad (\forall \varepsilon > 0)}$$

<u>Rmk:</u> Pf for $x \in \mathbb{Z}^d$ is sufficient. Only the RHS has a small difference.

<u>Lem</u> $\forall x \in \mathbb{Z}^d$, $\mathbb{P}[X = x] = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \phi_X(\theta) e^{-ix\cdot\theta} d\theta$,

where $\phi_X(\theta) = \mathbb{E}[e^{iX\cdot\theta}]$

Note the similarity with the Fourier Inversion Formula.

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{2\pi i x\cdot\xi} d\xi$$

The main difference is that we are only integrating over $[-\pi,\pi]^d$ instead of the whole domain $\mathbb{R}^d$.

This seems more like Fourier Series.

This comes from ~~that~~ the fact that $x \in \mathbb{Z}^d$.

<u>Prf</u> $\phi_X(\theta) = \phi_X(\theta + 2\pi k)$, for any $k \in \mathbb{Z}^d$.

That is, for integer-valued RVs, its characteristic function is periodic.

$$= \int_{\mathbb{R}^d} e^{i(\theta + 2\pi k)x} \mathbb{P}_X(dx)$$

$$= \int_{\mathbb{R}^d} e^{i\theta} \cdot \underbrace{e^{i2\pi \overbrace{k\cdot x}^{\in \mathbb{Z}}}}_{=1} \mathbb{P}_X(dx).$$

$$\text{RHS} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \left(\int_{\mathbb{R}^d} e^{i\theta\cdot y}\, \mathbb{P}_X(y) \right) e^{-i\theta\cdot x}\, d\theta$$

$$= \int_{\mathbb{R}^d} \left(\underbrace{\frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} e^{i\theta(y-x)}\, d\theta}_{\delta_x(y)} \right) \mathbb{P}_X(\cancel{dx}\ dy)$$

We used Fubini

$$= \int_{\mathbb{R}^d} \delta_x(y)\, \mathbb{P}_X(dy)$$

$$= \mathbb{P}[X = x]$$

$\blacksquare$

<u>Prf</u> (LCLT) Assume $d = 1$.

Let $I = \sigma n^{\frac{1}{2}} \left| \mathbb{P}[S_n = x] - \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left\{ -\frac{x^2}{2n\sigma^2} \right\} \right|$

$$= \cancel{\sigma n^{\frac{1}{2}}} \left| \frac{\sigma n^{\frac{1}{2}}}{2\pi} \int_{[-\pi,\pi]} \underbrace{\phi_{S_n}(\theta)}_{=(\phi_X(\theta))^n} e^{-ix\theta}\, d\theta - \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x^2}{2n\sigma^2} \right\} \right|$$

Let $u = \frac{x}{\sqrt{n}\sigma}$

$$= \left| \frac{\sigma n^{\frac{1}{2}}}{2\pi} \int_{[-\pi,\pi]} (\phi_X(\theta))^n e^{-i\sqrt{n}\sigma u\theta}\, d\theta - \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{u^2}{2} \right\} \right|$$

Let $s = \sqrt{n}\sigma\theta$

$$= \left| \frac{1}{2\pi} \int_{[-\sqrt{n}\sigma\pi, \sqrt{n}\sigma\pi]} \left(\phi_X\left(\frac{s}{\sqrt{n}\sigma} \right) \right)^n e^{-ius}\, ds - \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{u^2}{2} \right\} \right|$$

Notice that $\exp\left\{ -\frac{u^2}{2} \right\} = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} e^{-isu}\, ds$

$$= \left| \frac{1}{2\pi} \int_{\mathbb{R}} \left(\mathbb{1}_{[-\sqrt{n}\sigma\pi, \sqrt{n}\sigma\pi]} \left(\phi_X\left(\frac{s}{\sqrt{n}\sigma} \right) \right)^n - e^{-\frac{s^2}{2}} \right) e^{-isu}\, ds \right|$$

. Recall the proof of CLT, we say that

$$\left(\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right)^n \approx \left(1 - \frac{1}{2}\left(\frac{s}{\sqrt{n}\sigma}\right)^2 + \quad \cdots \quad \right)^n \approx e^{-\frac{s^2}{2}}$$

$$\frac{\left(i \cdot \frac{s}{\sqrt{n}\sigma}\right)^2}{2!} \mathbb{E}[X^2] = -\frac{1}{2} \frac{s^2}{n\sigma^2} \cdot \sigma^2 = -\frac{1}{2} \frac{s^2}{n}.$$

This expansion relies on s being small. But we are integrating over $s \in \mathbb{R}$. ~~Fortunately~~, we ~~have~~ have $\sqrt{n}$ in $\frac{s}{\sqrt{n}}$. So it's fine.

Sketch: $I = I_1 + I_2 + I_3 + I_4$

$$I_1 = \int_{[-n^{\varepsilon}\sigma_\pi,\ n^{\varepsilon}\sigma_\pi]} \left|\left(\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right)^n - e^{-\frac{s^2}{2}}\right| ds \qquad \varepsilon > 0$$

$$I_2 = \int_{n^{\varepsilon}\sigma_\pi \le |s| \le \varepsilon\sqrt{n}\sigma_\pi} \left|\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right|^n ds$$

mecroscopic

(micro < mecro < macro)

$$I_3 = \int_{\varepsilon\sqrt{n}\sigma \le |s| \le \sqrt{n}\sigma\gamma} \quad \cdots \quad ds$$

Microscopic

$$I_4 = \int_{|s| \ge n^{\varepsilon}\sigma_X} e^{-\frac{s^2}{2}} ds$$

Resumed in ~~next lecture~~ lecture 2.

$$I \le I_1 + I_2 + I_3 + I_4.$$

$$I_1 = \frac{1}{2\pi}\int_{[-n^{\varepsilon}\sigma x,\, n^{\varepsilon}\sigma x]} \left|\left(\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right)^n - e^{-\frac{s^2}{2}}\right| ds$$

Recall proof of CLT, we used

$\underline{\text{Lem}}$ $$\left|e^{ix} - \left(1 + ix - \frac{1}{2}x^2\right)\right| \le \min\left\{\frac{|x|^3}{3}, \frac{|x|^2}{2}\right\}$$

see Durrett

$$\Rightarrow \phi_X(\cancel{\tfrac{s}{\sqrt{n}\sigma}}\,\theta) = \mathbb{E}\left[e^{iX\theta}\right] \Rightarrow \left|\phi_X(\theta) - \left(1 - \frac{\sigma^2\theta^2}{2}\right)\right| \le \min\left\{\frac{\theta^3\mathbb{E}[X^3]}{3}, \frac{\theta^2\sigma^2}{2}\right\}$$

$$\Rightarrow \left|\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right) - \left(1 - \frac{s^2}{2n}\right)\right| \le \min\left\{\frac{s^3 C(\rho,\sigma)}{3n^{\frac{3}{2}}}, \frac{s^2 C(\rho,\sigma)}{2n}\right\} \le C\left(\frac{s}{\sqrt{n}}\right)^3$$

Notice the $n^{\frac{3}{2}}$. This cannot be ignored if s is ~~too~~ too large, which is why we limited $[-n^{\varepsilon}\sigma x, n^{\varepsilon}\sigma x]$.

$$\left|\left(\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right)^n - e^{-\frac{s^2}{2}}\right| \le \left|\left(1 - \frac{s^2}{2n} + C\left(\frac{s}{\sqrt{n}}\right)^3\right)^n - e^{-\frac{s^2}{2}}\right| \le c\left(\frac{s}{\sqrt{n}}\right)^3$$

$$= \left|e^{n\log\left(1 - \frac{s^2}{2n} + C\left(\frac{|s|}{\sqrt{n}}\right)^3\right)} - e^{\frac{s^2}{2}}\right|$$

$\boxed{x>0,\ \log(1+x)\le x}$

$$\le \left|e^{n\left(-\frac{s^2}{2n} + C\left(\frac{|s|}{\sqrt{n}}\right)^3\right)} - e^{\frac{s^2}{2}}\right|$$

$$= e^{-\frac{s^2}{2}}\left|e^{C\frac{|s|^3}{\sqrt{n}}} - 1\right|$$

$$\le C\frac{|s|^3}{\sqrt{n}} \le Cn^{-\frac{1}{2}+3\varepsilon} \quad (\text{since } |s| < n^{\varepsilon}\sigma x)$$

$$I_1 \le Cn^{-\frac{1}{2}+4\varepsilon}$$ (the extra ε is from integrating $Cn^{-\frac{1}{2}+3\varepsilon}$ over $[-n^{\varepsilon}\sigma x, n^{\varepsilon}\sigma x]$. So $\int_{[\ ]} Cn^{-\frac{1}{2}+3\varepsilon} = C\,2n^{\varepsilon}\sigma x\, n^{-\frac{1}{2}+3\varepsilon}$

$$= O(n^{-\frac{1}{2}+4\varepsilon}).$$

$$\therefore I_2 = \frac{1}{2\pi}\int_{\{|s| \geq n^{\varepsilon}\sigma x\}} e^{-\frac{s^2}{2}} ds$$

Lem

$$\int_x^{+\infty} e^{-\frac{y^2}{2}} dy \leq \frac{1}{x} e^{-\frac{x^2}{2}}$$

Prf $y = x+z$, $dy = dz$

$$\text{LHS} = \int_0^{+\infty} e^{-\frac{(x+z)^2}{2}} dz$$

$$= \int_0^{+\infty} e^{-\left(\frac{1}{2}x^2 + \frac{z^2}{2}\right) - xz} dz$$

$$\leq e^{-\frac{1}{2}x^2}\int_0^{+\infty} e^{-xz} dz$$

$$= \frac{1}{x} e^{-\frac{x^2}{2}}.$$

$\blacksquare$

$$I_2 = \frac{1}{2\pi}\left(\int_{n^{\varepsilon}\sigma x}^{+\infty} + \int_{-\infty}^{-n^{\varepsilon}\sigma x}\right)\left(e^{-\frac{y^2}{2}} dy\right)$$

$$\leq \frac{1}{2\pi}\left(\frac{1}{n^{\varepsilon}\sigma x} e^{-\frac{(n^{\varepsilon}\sigma x)^2}{2}}\right)\cdot 2$$

$$= \frac{1}{\pi n^{\varepsilon}\sigma x} e^{-\frac{(n^{\varepsilon}\sigma x)^2}{2}}$$

$$\ll n^{-\frac{1}{2}} \quad (n \to +\infty)$$

$$I_3 = \frac{1}{2\pi}\int_{n^{\varepsilon}\sigma\pi \leq |s| \leq \varepsilon\sqrt{n}\sigma\pi} \left(\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right)^n ds$$

Why isn't the ~~bottom~~ interval over which we integrate just $\{n^{\varepsilon}\sigma\pi \leq |s| \leq \sqrt{n}\sigma\pi\}$? Let's calculate to find out.

$$\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right) \simeq 1 - \frac{s^2}{2n} + \min\left\{ c_1 \left(\frac{|s|}{\sqrt{n}}\right)^2, \; c_2\left(\frac{|s|}{\sqrt{n}}\right)^3 \right\}$$

if $\frac{s}{\sqrt{n}} \ll 1$, then $\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right) \le 1 - \frac{s^2}{4n}$

(Taylor series)

$$\left(\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right)^n \le \left(1 - \frac{s^2}{4n}\right)^n \le e^{-\frac{s^2}{4}}$$

We can then apply the above lemma to get result.

This ~~allow~~ requires small enough s. So if simply

$|s| \le \varepsilon\sqrt{n}\sigma_X$, then $\frac{|s|}{\sqrt{n}} \le \sigma_X$ not $\ll 1$.

Since we have $\varepsilon\sqrt{n}\sigma_X$, $\frac{|s|}{\sqrt{n}} \ll \varepsilon\sigma_X$, so the size of $\frac{s}{\sqrt{n}}$ can be controlled.

$$I_3 \le \frac{1}{2\pi}\int_{n^{\varepsilon}\sigma_X \le |s| \le \varepsilon\sqrt{n}\sigma_X} e^{-\frac{s^2}{4}}\,ds$$

$$\le \frac{1}{2\pi}\left(C e^{-C n^{2\varepsilon}(\sigma_X)^2} + C e^{-\varepsilon^2 n(\sigma_X)^2} \right) \ll n^{-\frac{1}{2}}.$$

Finally

$$I_4 = \frac{1}{2\pi}\int_{\varepsilon\sqrt{n}\sigma_X \le |s| \le \sqrt{n}\sigma\varepsilon} \left(\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right)^n ds$$

Method for I_3 doesn't work because no $\frac{s}{\sqrt{n}} \ll 1$.

Method for I_1 doesn't work because we are not comparing $\left(\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right)^n$ with anything.

Notice that we haven't invoked aperiodicity.

How this ~~ap~~ helps is not so obvious.

$\therefore$ By aperiodicity, on the interval over which we integrate,

$$\left|\phi_X\left(\frac{s}{\sqrt{n}\sigma}\right)\right| \le e^{-C_\varepsilon} \quad \text{(not rigorous)}$$

So $I_4 \le (1-\varepsilon)\sigma\pi\sqrt{n}\, e^{-nC_\varepsilon} \ll n^{-\frac{1}{2}}$ as $n \to +\infty$.

Put the above ~~together~~.

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Applications of ~~LCDT~~ LCLT.

Recurrence v.s. Transience.

Def $\tau_R(x) =$ R-th time to visit x.

~~Recurrent~~ $\rho_{x,y} = \mathbb{P}_x[\tau_1(y) < +\infty]$

x is Recurrent $\iff \rho_{x,x} = 1$

transient $\iff \rho_{x,x} < 1$.

Cor x is recurrent

$\implies \mathbb{P}_x(\text{i.o. } S_n = x) = 1$.

Cor $N_x = \#$ visits to x.

$\mathbb{E}_x[N_x] = +\infty \iff$ recurrent

$\mathbb{E}_y[N_x] = -\infty \iff$ transient

Thm RW on $\mathbb{Z}^d$, centred X, $\mathrm{Var}[X] = \sigma^2$, irreducible, aperiodic $\mathbb{E}[|X|^3] < \infty$, then it is recurrent for $d=1,2$, and transient for $d \ge 3$.

Pf Suffices to compute $\sum_{n=1}^{+\infty} \mathbb{P}[S_n = 0]$ by the above corollary. Because we assumed irreducibility and aperiodicity, we only need to show the recurrence / transience of 0.

$$\mathbb{P}[S_n = 0] \asymp \frac{1}{(\sqrt{2\pi n\sigma^2})^{\frac{d}{2}}} + O\left(n^{-\frac{d}{2}-\frac{1}{2}+\varepsilon}\right)$$

$$\asymp n^{-\frac{d}{2}}\left(1 + n^{-\frac{1}{2}+\varepsilon}\right)$$

$$\asymp n^{-\frac{d}{2}}$$

Recall that $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ converges iff $p = 1, 2$.

We are done.

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We have put a lot of assumptions on the RW to apply the ~~4~~ LCLT.

<u>Thm</u> What happens if variance does not exist? (Chung-Fuchs)

RW is recurrent iff. (applicable for $\mathbb{R}^d$ as well)

$$\lim_{r \uparrow 1} \int_{[-\pi,\pi]^d} \mathrm{Re}\left(\frac{1}{1-r\phi_X(\theta)}\right) d\theta = +\infty$$

Pf (For $\mathbb{Z}^d$ only)

Again, we compute $\sum_{n=1}^{+\infty} \mathbb{P}[S_n=0]$. ~~4~~ $= \lim_{r\nearrow 1} \sum_{n=0}^{+\infty} \mathbb{P}[S_n=0] r^n$.

$$= \lim_{r\nearrow 1} \sum_{n=0}^{+\infty} \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \phi_{S_n}(\theta) e^{-i\theta\cdot 0} d\theta \; r^n$$

$$= \lim_{r\nearrow 1} \sum_{n=0}^{+\infty} \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} (r\phi_X(\theta))^n d\theta$$

(Fubini)

$$= \lim_{r\uparrow 1} \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \sum_{n=0}^{+\infty} (r\phi_X(\theta))^n d\theta.$$

$$= \lim_{r\uparrow 1} \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{1}{1-r\phi_X(\theta)} d\theta$$

Other Estimates

<u>Thm</u> (Berry-Essen) X centred, $\mathrm{Var}[X]=\sigma^2$, $\mathbb{E}[|X|^3] = \rho < +\infty$.

Let $F_n(x) = \mathbb{P}\left[\frac{S_n}{\sqrt{n}\sigma} \leq x\right]$, $\Phi(x)$ the $\mathcal{N}(0,1)$ distribution function.

$$|F_n(x) - \Phi(x)| \leq \frac{3\rho}{\sqrt{n}}$$

Stein Method

We have 2 measures ~~RVs~~, so how do we describe their 'distance'?

$d(\mathbb{P},\mathbb{Q}) := \sup_{f\in D} \left\{ \int_{\mathbb{R}} f\, d\mathbb{P} - \int_{\mathbb{R}} f\, d\mathbb{Q} \right\}$, for some dictionary ~~some test function~~ $f \in D$ (~~distribution~~) ~~dictionary~~

$D :=$ bounded $\|f\|_{L^\infty} \leq 1 \longrightarrow$ Total variation

$D := 1$-Lip. $\Rightarrow$ Wasserstein

Thm $d_{Wasserstein}\left(\frac{S_n}{\sqrt{n}\sigma}, \mathcal{N}(0,1)\right) \leq \frac{3\rho}{\sqrt{n}}$.