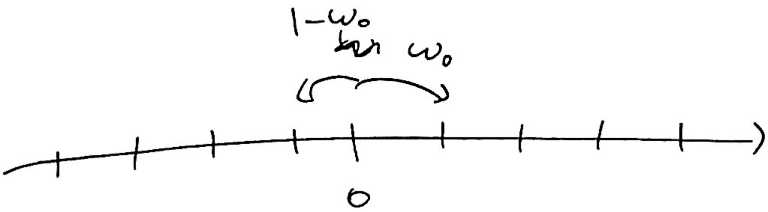
Lecture 3 RWRE

1D RWRE

"solvable" model



Random environment: $(\omega_x)_{x\in\mathbb{Z}} \sim \mathbb{P}$

$\omega_x \in [c, 1-c]$, $c \in (0,1)$

assume ω_x iid.

Random walk: Given a fixed $\omega = (\omega_x)$, we ~~ax~~ define a 1D RW measure P^{ω}, where (1) $P^{\omega}[X_{n+1} = x+1 \mid X_n = x, X_{n-1}, \dots, X_0]$

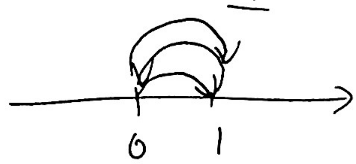
$= P^{\omega}[X_{n+1} = x+1 \mid X_n = x] = \omega_x$ (Markov property)

(2) $P^{\omega}[X_{n+1} = x-1 \mid X_n = x] = 1-\omega_x$

Remark: (X_n) is Markov under P^{ω}, but not under $\mathbb{P}$.

Clarification: $\mathbb{P}[(X_n)\in A] := \mathbb{E}\left[P^{\omega}[(X_n)\in A]\right]$.

Counterexample: $\mathbb{P}[X_1 = 1, X_2 = 0, X_3 = 1]$



$$\mathbb{P}[X_3 = 1 \mid X_2 = 0, X_1 = 1] \overset{?}{=} \mathbb{P}[X_3 = 1 \mid X_2 = 0].$$

Calculation:

$$\text{LHS} = \frac{\mathbb{P}[X_3=1, X_2=0, X_1=1]}{\mathbb{P}[X_2=0, X_1=1]} = \frac{\mathbb{E}[P^{\omega}[X_3=1, X_2=0, X_1=1]]}{\mathbb{E}[P^{\omega}[X_2=0, X_1=1]]}$$

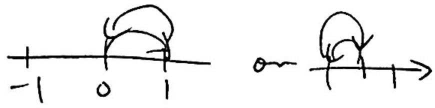
$$= \frac{\mathbb{E}[\omega_0^2(1-\omega_1)]}{\mathbb{E}[\omega_0(1-\omega_1)]} = \frac{\mathbb{E}[\omega_0^2]\,\mathbb{E}[1-\omega_1]}{\mathbb{E}[\omega_0]\,\mathbb{E}[1-\omega_1]} \quad \text{(indep)}$$

$$= \frac{\mathbb{E}[\omega_0^2]}{\mathbb{E}[\omega_0]}$$

$$\text{RHS} = \frac{\mathbb{E}[\mathbb{P}^\omega[X_2=0, X_3=1]]}{\mathbb{E}[\mathbb{P}^\omega[X_2=0]]} = \frac{\mathbb{E}[\omega_0 \mathbb{P}^\omega[X_2=0]]}{\mathbb{E}[\mathbb{P}^\omega[X_2=0]]}$$

$$= \frac{\mathbb{E}[\omega_0^2(1-\omega_1) + \omega_0(1-\omega_0)\omega_{-1}]}{\mathbb{E}[\omega_0(1-\omega_1) + (1-\omega_0)\omega_{-1}]}$$

$X_2 = 0$:



$$\mathbb{P}^\omega[X_2=0] = \omega_0(1-\omega_1) + (1-\omega_0)\omega_{-1}$$

$$= \frac{\mathbb{E}[\omega_0^2]\,\mathbb{E}[1-\omega_1] + \mathbb{E}[\omega_0-\omega_0^2]\,\mathbb{E}[\omega_{-1}]}{\mathbb{E}[\omega_0]\,\mathbb{E}[1-\omega_1] + \mathbb{E}[1-\omega_0]\,\mathbb{E}[\omega_{-1}]}$$

$$= \frac{\mathbb{E}[\omega_0^2](1-\mathbb{E}[\omega_0]) + (\mathbb{E}[\omega_0]-\mathbb{E}[\omega_0^2])\,\mathbb{E}[\omega_0]}{\mathbb{E}[\omega_0](1-\mathbb{E}[\omega_0]) + (1-\mathbb{E}[\omega_0])\,\mathbb{E}[\omega_0]}$$

$$\cancel{= \frac{1-\mathbb{E}[\omega_0] + \mathbb{E}[\omega_0^2] - \mathbb{E}[\omega_0^2]}{2\mathbb{E}[\omega_0](1-\mathbb{E}[\omega_0])} = \frac{1-\mathbb{E}[\omega_0^2]}{2(1-\mathbb{E}[\omega_0])}}$$

$$= \frac{\mathbb{E}[\omega_0^2] - 2\mathbb{E}[\omega_0^2]\,\mathbb{E}[\omega_0] + (\mathbb{E}[\omega_0])^2}{2\mathbb{E}[\omega_0](1-\mathbb{E}[\omega_0])}$$

$$\overset{?}{=} \frac{\mathbb{E}[\omega_0^2]}{\mathbb{E}[\omega_0]} = \text{LHS}$$ Not equal.

This is because past history gives information about the environment. So not Markov in $\mathbb{P}$.

Recurrence / Transience?

We can answer for 1D RWRE.

Perhaps depends on $\mathbb{E}[\omega_0 - (1-\omega_0)]$? ~~speed~~ $(= \mathbb{E}[2\omega_0 - 1])$

If $\mathbb{E}[\omega_0 - (1-\omega_0)] > 0 \Rightarrow$ drift to right? $\Rightarrow$ transient

Not exactly.

$\underline{\text{Thm}}$ let $\rho_x = \dfrac{1-\omega_x}{\omega_x}$

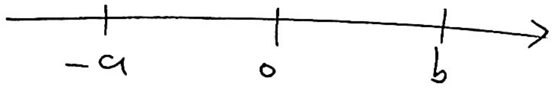
$$\mathbb{E}[\log \rho_x] = \begin{cases} <0 & \lim\limits_{n\to\infty} X_n = +\infty \\ >0 & \lim\limits_{n\to\infty} X_n = -\infty \\ =0 & \limsup\limits_{n\to\infty} X_n = +\infty \\ & \liminf\limits_{n\to\infty} X_n = -\infty \end{cases}$$

(oscillatory).

$\longrightarrow$ Related to Gambler's Ruin.

$\underline{\text{Prf}}$: Define hitting time $\tau_{\cancel{x}n} = \inf\{k \in \mathbb{N}_{\cancel{\ }},\ X_k = \cancel{x}_n\}$ of $\cancel{x}_n$.
$k \geq 0$

$$\mathbb{P}^{\omega}_{\underset{\uparrow}{x}}[\tau_b < \tau_{-a}] =: h(x)$$



start at x Boundary conditions:

$$h(b) = 1,\ h(a) = 0$$

$$\forall x \in (-a,b)\ h(x) = \mathbb{P}^{\omega}_{x}[\tau_b < \tau_{-a},\ X_1 = x+1] + \mathbb{P}^{\omega}_{x}[\cancel{\ }\tau_b \cancel{>} < \tau_{-a},\ X = x-1$$

(first step)

$$h(x) = \mathbb{P}^{\omega}_{x+1}(\tau_b < \tau_{-a}]\ \omega_x + \mathbb{P}^{\omega}_{x-1}[\tau_b < \tau_{-a}]\ (1-\omega_x)$$

$$h(x) = \omega_x h(x+1) + (1-\omega_x) h(x-1)$$

Fortunately, we are in 1D so we have a recurrence relation.

$$h(x+1) = \frac{1}{\omega_x} h(x) - \frac{1-\omega_x}{\omega_x} h(x-1)$$

$$= \frac{1}{\omega_x} h(x) - \rho_x h(x-1)$$

$$= h(x) + \rho_x\big(h(x) - h(x-1)\big)$$

Define $\Delta h(x) = h(x) - h(x-1)$

Then $\Delta h(x+1) = \rho_x \Delta h(x)$. for $\cancel{x}-a+1 \leq x \leq \cancel{b-1}\ b$

So $\Delta h(x) = \rho_{x-1} \Delta h(x-1)$

$$= \rho_{x-1} \rho_{x-2} \Delta h(x-2)$$

$$= \prod_{j=-a+1}^{x-1} \rho_j \, \Delta h(-a+1)$$

$$= \prod_{j=-a+1}^{x-1} \rho_j \left(h(-a+1) - h(a) \right)$$

$$= \left(\prod_{j=-a+1}^{x-1} \rho_j \right) h(-a+1)$$

$$h(x) = \sum_{k=-a+1}^{x} \Delta h(k)$$

$$\forall x \in (-a,b), \quad h(x) = \sum_{j=-a}^{x} \left(\prod_{j=a+1}^{k} \rho_j \right) \Delta h(-a+1)$$

$$\cancel{\Delta h(b) = \prod_{j=-a+1}^{b} \rho_j \, \Delta h(-a+1)}$$

then $h(x) = \Delta h(-a+1) \sum_{j=-a}^{x} \left(\prod_{j=a+1}^{k} \rho_j \right)$

$$1 = h(b) = \sum_{j=-a}^{b} \left(\prod_{j=a+1}^{k} \rho_j \right) \cdot \Delta h(-a+1)$$

so $h(x) = \dfrac{\sum_{j=-a}^{x} \left(\prod_{j=a+1}^{k} \rho_j \right)}{\sum_{j=-a}^{b} \left(\prod_{j=a+1}^{k} \rho_j \right)}$

we can ~~see~~ start to see the leg here.

$$\mathbb{P}_0^{\omega}[\tau_b < +\infty] = \lim_{a \to +\infty} \mathbb{P}_0^{\omega}[\tau_b < \tau_{-a}]$$

$$= \lim_{a \to +\infty} \frac{\sum_{k=-a+1}^{0} \left(\prod_{j=k-a+1}^{b} \rho_j \right)}{\sum_{k=-a+1}^{0} \left(\prod \quad \right) + \underbrace{\sum_{k=1}^{b} \left(\prod \quad \right)}_{\cancel{\text{fixed under } a \to +\infty}}}$$

only a varies. b fixed. we remove the a dependence from this term

$$= \lim_{a \to +\infty} \frac{\sum_{k=a+1}^{0} \left(\prod_{j=k+1}^{0} \rho_j \right)^{-1}}{\sum_{k=a+1}^{0} \left(\prod_{j=k+1}^{0} \rho_j \right)^{-1} + \underbrace{\sum_{k=1}^{b} \left(\prod_{j=1}^{b} \rho_j \right)}_{\text{constant in } a.}}$$

$$\rho_1 \rho_2 \cdots \rho_n = \exp\left(\sum_{j=1}^{n} \log \rho_j \right)$$

If $\mathbb{E}[\log \rho_x] < 0$, then $\simeq \exp(-n\varepsilon)$ a.s. by LLN.

So if $\mathbb{E}[\log \rho_x] < 0$, then $\sum_{k=a+1}^{0} \left(\prod_{j=k+1}^{0} \rho_j \right)^{-1} \to +\infty$

So $\lim_{a \to +\infty} \mathbb{P}_0^{\omega} [\tau_b < \tau_{-a}] = 1$.

~~$\mathbb{E}[\log \rho_x]$ sus~~ $\log \rho_x = \log(1 - \omega_x) - \log \omega_x$

$\log \rho_x < 0 \Rightarrow \log(1-\omega_x) < \log \omega_x$

$1 - \omega_x < \omega_x$

So $\mathbb{E}[\log \rho_x] < 0$ means drift right.

So $\forall b > 0$, $\mathbb{P}^{\omega}[\tau_b < +\infty] = 1$ makes sense.

Exercise: Show for <0 case.

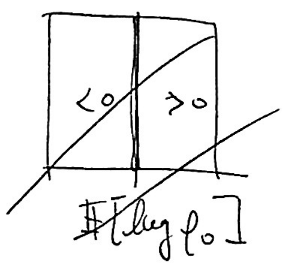
Last case ($=0$): $\prod_{k=-a+1}^{0}\left(\prod_{j=k+1}^{0}\rho_j\right)^{-1} \sim$ constant ???

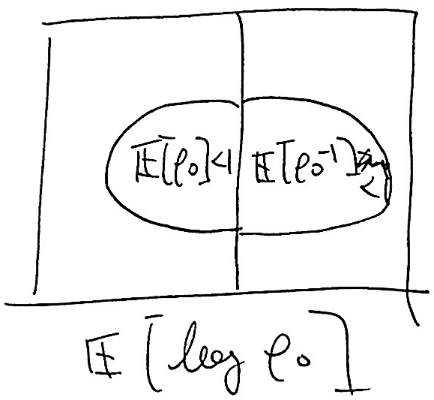
another exercise. $\square$

LLN? rate of convergence?

Thm
$$\lim_{n\to+\infty}\frac{X_n}{n} = \begin{cases} \dfrac{1-\mathbb{E}[\rho_0]}{1+\mathbb{E}[\rho_0]} & \mathbb{E}[\rho_0]<1 \\ 0 & \mathbb{E}[\rho_0]\geq 1,\ \mathbb{E}[\rho_0^{-1}]\geq 1 \\ \dfrac{1-\mathbb{E}[\rho_0^{-1}]}{1+\mathbb{E}[\rho_0^{-1}]} & \mathbb{E}[\rho_0^{-1}]<1 \end{cases}$$

Compare $\mathbb{E}[\log \rho_0]$ and $\mathbb{E}[\rho_0]$, $\mathbb{E}[\rho_0^{-1}]$





$$\mathbb{E}[\log\rho_0] = \mathbb{E}[\log(1+(\rho_0-1))] \leq \mathbb{E}[\rho_0-1]$$

ω_x large $\Rightarrow \rho_x$ small.

So if $\mathbb{E}[\rho_0]<1$, $\mathbb{E}[\log\rho_0]<0$.

$\mathbb{E}[\log\rho_0]>0$ sensitive to right drift.

$\mathbb{E}[\rho_0]$ need enough right drift to <1.

Proof: Not like usual LLN: no sum of iid RV for RWRE.

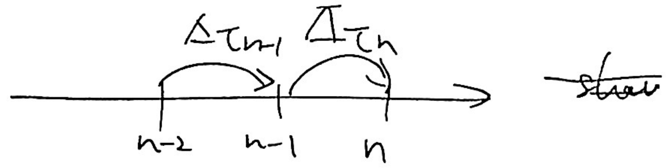
We will make use of hitting time $T_n = \inf\{k\in\mathbb{N},\ X_k = n\}$

Let $\Delta T_n := T_n - T_{n-1}$, after hit $n-1$, how long until n?

~~Observation: this makes~~ $\Delta T_n \geq 1$ in 1D. for $n>0$, start at 0

Observation: $\lim_{n\to+\infty}\frac{T_n}{n} = \mathbb{E}[\Delta T_1]$ $\left(= \mathbb{E}[\mathbb{E}^{\omega}(\Delta T_1)]\right)$

• How to observe?



$\tau_n = \sum \Delta\tau_j$. ~~st~~ $\Delta\tau_j$ should be identically distributed. ~~Is it~~ Are they ~~to~~ independent? Perhaps not.

[Birkhoff's thm Let T be measure-preserving transformation, ergodic, then

$$\frac{1}{n}\sum_{k=0}^{n} T^k \circ f \xrightarrow{a.s.} \int_\Omega f \, d\mathbb{P}.$$

$\Omega = (\omega_x)_{x\in\mathbb{Z}}$. $T : \Omega \to \Omega$: shift.

$\cancel{T(\dots, \omega_{-1}, \omega_0, \omega_1, \dots)}$

~~M~~: $\mathbb{P}$: iid (ω_x)

$\cancel{f := f(\dots) = \Delta\tau_{k+1}}$

$f := \Delta\tau_1$ (depended on (ω_x)))

$\Delta\tau_2 = T\circ f = T\circ \Delta\tau_1$

So $\frac{1}{n}\sum_{k=0}^{n} \Delta\tau_n \xrightarrow{a.s.} \mathbb{E}[\Delta\tau_1]$. ~~a.s.~~

~~second step~~:

Step 2: $X_n^* = \max_{k\le n} X_k$, we have

$\tau_{X_n^*} \le n$, by definition.

$< \tau_{X_n^*+1}$ (~~not~~ $\neq \tau_{X^*_{n+1}}$)

~~Ehh~~ Now consider $\mathbb{E}[\log\rho_0] < 0$ case. Then $\lim_{n\to\infty} X_n = +\infty$

$$\Rightarrow \frac{\tau_{X_n^*}}{X_n^*} \le \frac{n}{X_n^*} < \frac{\tau_{X_n^*+1}}{X_n^*+1} \, \frac{X_n^*+1}{X_n^*+1}$$

Let $n \to +\infty$, then $X_n^* \to +\infty$ and

$$\mathbb{E}[\Delta\tau_1] \le \lim_{n\to+\infty} \frac{n}{X_n^*} \le \mathbb{E}[\Delta\tau_1]$$

So $\displaystyle \lim_{n\to\infty} \frac{X_n^*}{n} = \frac{1}{\mathbb{E}[\Delta\tau_1]}$.

Step 3 : ~~general case~~.

$$\lim_{n\to+\infty} \frac{X_n}{n} = \lim_{n\to\infty} \frac{X_n^*}{n} \pm \frac{|X_n^* - X_n|}{n}$$

$$|X_n^* - X_n| \le n - \tau_{X_n^*} \quad \cancel{\text{worst case}}$$

Why? worst case:

$$|X_n^* - X_n| \le |\tau_{X_n^*} - n| = n - \tau_{X_n^*}$$

$|\text{space}| \le |\text{time}|$.

So $$\frac{|X_n^* - X_n|}{n} \le \frac{n - \tau_{X_n^*}}{n} = 1 - \frac{\tau_{X_n^*}}{n}$$

$$= 1 - \frac{\tau_{X_n^*}}{X_n^*} \, \frac{X_n^*}{n}$$

$$\longrightarrow 1 - \mathbb{E}[\Delta\tau_1] \cdot \frac{1}{\mathbb{E}[\Delta\tau_1]} = 0.$$

南 Exercise: Calculate $\mathbb{E}[\Delta\tau_1]$ to obtain final conclusion.

Hint: Use Markov property to get recurrence relations.

What if $\mathbb{E}[\Delta\tau_1] = +\infty$ $\left(\lim_{n\to+\infty} \frac{X_n}{n} = 0\right)$

(so slightly drift right)

Is the scaling of $\frac{1}{n}$ for X_n too large?

Thm (Kesten-Kozlov-Spitzer), (A limit law for RW in RE)

$$\lim_{n\to+\infty} \frac{\log X_n}{\log n} = \cancel{k} K \text{ for } X_n \simeq n^K, \text{ where } K\in(0,1)$$

How to use Birkhoff's ergodic Theorem on $\Delta\tau_k$?

$\Delta\tau_k$ is random not only in $(\omega_x)_{x\in\mathbb{Z}}$, cannot use ergodicity of T on $(\omega_x)_{x\in\mathbb{Z}}$.

We will construct an integrated ~~Prob~~ prob space.

indep. $\begin{cases} (\omega_x) \text{ iid} \\ (\vec{\xi}_k)_{k\in\mathbb{N}} \quad \vec{\xi}_k = (\xi_{k,j})_{j\in\mathbb{N}} \quad \text{iid } \xi_{k,j} \sim \mathrm{Unif}[0,1] \end{cases}$

Divide (X_n) into segments on $[\tau_{k-1}, \tau_k)$.

$$X_{n+1} = X_n + \mathbb{1}_{\{\xi_{k,j} \le \omega_{X_n}\}} - \mathbb{1}_{\{\xi_{k,j} > \omega_{X_n}\}}, \text{ where}$$

$\tau_{k-1} \le n < \tau_k$, $j = n - \tau_{k-1}$.

Prop. Under $\mathbb{P}^\omega$, $\Delta\tau_k$ are a independent.

$$\Delta\tau_1 = f\left((\omega_x), \vec{\xi}\right)$$

$$\Delta\tau_2 = T \circ f\left((\omega_x), \vec{\xi}\right) = f\left(T(\omega, \vec{\xi})\right)$$

Let $\Omega = \left\{ \left((\omega_x)_{x\in\mathbb{Z}}, (\vec{\xi}_k)_{k\in\mathbb{N}} \right) \right.$

$T : \Omega \to \Omega$ shift:

$$\left(T(\omega, \vec{\xi})\right)_{x,k} := \left(\omega_{x+1}, \vec{\xi}_{k+1}\right).$$

$\Big\}$ Q - e

T is measure-preserving

$$\lim_{n\to\infty} \frac{1}{n} \sum \Delta\tau_k = \lim_{n\to\infty} \frac{1}{n}\tau_n = \mathbb{E}[\Delta\tau_1] \quad Q \text{ a.s.}$$

$\Rightarrow$ $\mathbb{P}$ a.s. (Why? because ~~Q~~ $\mathbb{P}$ is a projection of Q