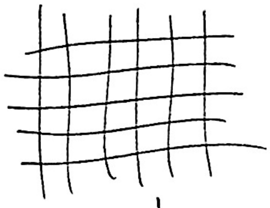
Lecture 4 Random Conductance Model.



$\mathbb{Z}^d$

$\omega : E^d \to [\Lambda^{-1}, \Lambda]$ $\quad \Lambda \geq 1.$

i.i.d. r.v. $\quad \omega_{\cancel{x}} \sim$ ~~conduction~~ conductance $= \dfrac{1}{\text{resistance}}$

Random walk: (Markov Jump Process)

VSRW $\quad \cancel{\ }\, \mathcal{L}_V f(x) = \sum_{y \sim x} \omega_{xy}\big(f(y) - f(x)\big)$ (variable speed)

CSRW $\quad \mathcal{L}_C f(x) = \sum_{y \sim x} \dfrac{\omega_{xy}}{\pi(x)}\big(f(y) - f(x)\big)$ (constant speed)

$$\pi(x) := \sum_{y \sim x} \omega_{xy}$$ (ω_{xy} not necessarily ω_{yx} for definition)

[Exercise: $\xi_1 \sim \mathcal{E}xp(\alpha_1)$, $\xi_2 \sim \mathcal{E}xp(\alpha_2)$, indep

$$\xi_1 \wedge \xi_2 \sim \mathcal{E}xp(\alpha_1 + \alpha_2)$$

$$\mathbb{P}[\xi_1 \wedge \xi_2 = \xi_1] = \frac{\alpha_1}{\alpha_1 + \alpha_2}$$

$$\mathbb{P}[\xi_1 \wedge \xi_2 = \xi_2] = \frac{\alpha_2}{\alpha_1 + \alpha_2}.$$

$\tau_k :=$ time of k-th jump.

$Y_k = X_{\tau_k}$

Then (Y_k) is same for both VSRW and CSRW.

$$\mathbb{P}^\omega[Y_{k+1} = y \mid Y_k = x] = \frac{\omega_{xy}}{\pi(x)}$$

~~Properties~~

Similar to SRW: $\omega \in [\lambda^{-1}, \lambda]$, $\lambda \geq 1 \Rightarrow$ controlled behaviour.

Recurrence / Transience

Electric network method. $\longleftrightarrow$ RW

$$\tau_x = \inf\{k \in \mathbb{N}_0 : Y_k = x\} \qquad \tau_x^+ = \inf\{k \in \mathbb{N}_{\geq 1} : Y_k = x\}$$

$$h(x) = \mathbb{P}_x^{\omega}[\tau_z < \tau_a] \quad \left(\text{to study } \mathbb{P}_a^{\omega}[\tau_z^+ < \tau_a]\right)$$

$$= \sum_{y \sim x} \mathbb{P}_x^{\omega}[\tau_z < \tau_a, Y_1 = y]$$

$$= \sum_{y \sim x} \mathbb{P}_x^{\omega}[\tau_z < \tau_a \mid Y_1 = y] \frac{\omega_{xy}}{\pi(x)}$$

$$= \sum_{y \sim x} \frac{\omega_{xy}}{\pi(x)} h(y)$$

$\Rightarrow \mathcal{L}_c h(x) = 0. \leftarrow$ ~~heat~~ Laplace equation.

Boundary conditions: $h(z) = 1$, $h(a) = 0$.

Dirichlet Problem (special version)

Prop Let $G = (V, E)$ be a finite graph, connected.

$$\begin{cases} \mathcal{L}_c h = 0 & V \setminus \{a, z\} \\ h(a) = g(a) \\ h(z) = g(z) \end{cases}$$

has unique solution

Prf Existence: already done:

$$h(x) = \mathbb{E}^{\omega}[g(\mathbb{1}_{\tau_a \wedge \tau_z})]$$

Uniqueness: maximum principle

~~Spoilers~~: effective resistance $R_{eff}(a \leftrightarrow z) = \dfrac{h(z) - h(a)}{\|\theta\|}$

current intensity $\|\theta\| := \sum_{y \sim a} \omega_{xy}\big(\underbrace{h(y)}_{I(\vec{a,y})} - \underbrace{h(a)}_{=0}\big)$

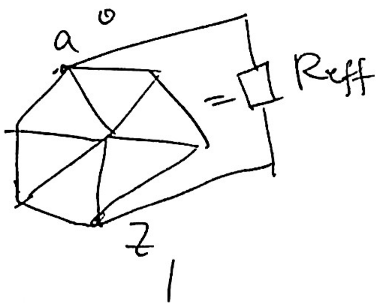
$\cancel{\mathbb{P}^{\omega}[\tau_a^+ < \tau_z]}$

$$\mathbb{P}^{\omega}_a[\tau_\infty < \tau_a^+] = \frac{1}{\pi(a)\, R_{eff}(a \leftrightarrow \infty)}$$

If $R_{eff}(a \leftrightarrow \infty) = +\infty$, then $\mathbb{P}^{\omega}[\tau_\infty < \tau_a^+] = 0$, i.e. $\tau_\infty \geq \tau_a^+$ a.s. So returns a.s., i.e. recurrent.

If $R_{eff}(a \leftrightarrow \infty) < +\infty$, then transient.

Prop $\mathbb{P}^{\omega}_a[\tau_a^+ > \tau_z] = \dfrac{1}{\pi(a) R_{eff}(a \leftrightarrow z)}$



Prf

Note: $\mathcal{L}_c h(x) = 0 \iff \sum_{y \sim x} I_{\vec{xy}} = 0$.

$\mathbb{P}^{\omega}_a[\tau_a^+ > \tau_z]$

$= \sum_{y \sim x} \mathbb{P}^{\omega}_a[\tau_a^+ > \tau_z \mid Y_1 = y]\, \mathbb{P}^{\omega}_a[Y_1 = y]$

$= \sum_{y \sim x} \mathbb{P}^{\omega}_y[\tau_a^+ > \tau_z] \cancel{\pi} \dfrac{\omega_{xy}}{\pi(a)} = \sum_{y \sim x} \mathbb{P}^{\omega}_y[\tau_a > \tau_z] \dfrac{\omega_{xy}}{\pi(a)}$

$$= \sum_{y\sim x} h(y) \frac{w_{xy}}{\pi(a)} = \frac{1}{\pi(a)} \sum_{y\sim x} \cancel{\pi(x)} \, w_{xy}\big(h(y)-h(a)\big)$$

$$= \frac{\|\theta\|}{\pi(a)} = \frac{\|\theta\|}{\underbrace{h(z)-h(a)}_{=1}} \cdot \frac{1}{\pi(a)}$$

$$= \frac{1}{R_{eff}(a \leftrightarrow z)\,\pi(a)} .$$

How to calculate R_{eff}?

Prop

$$\left[R_{eff}(a \leftrightarrow z)\right]^{-1} = \inf_{\substack{f(a)=0\\ f(z)=1}} \mathcal{E}(f) \quad \left(= \mathcal{E}(h)\right), \; \|\theta\| = \mathcal{E}(h)$$

where $\mathcal{E}(f)$ is the Dirichlet energy

$$\mathcal{E}(f) = \sum_{e\in E} \big(f(e^*) - f(e_*)\big)^2 w_e$$

$$= \frac{1}{2}\sum_{x\in V}\sum_{y\sim x} \big(f(x)-f(y)\big)^2 w_{xy} \quad (\text{need reversible, } w_{xy} = w_{yx})$$

Prf ① $h = \underset{\substack{f(a)=0\\ f(z)=1}}{\arg\min}\, \mathcal{E}(f) \quad \begin{pmatrix} \mathcal{L}_c h = 0 & V\setminus\{a,z\} \\ h(a)=0 & \\ h(z)=1 & \end{pmatrix}$

Let $\phi = 0$ on $\{a,z\}$, $\tilde h$ be the minimiser.

Then $\tilde h$ satisfies $\mathcal{E}(\tilde h) \le \mathcal{E}(\underbrace{\tilde h + \delta\phi}_{\text{still satisfies boundary conditions}})$

$$\mathcal{E}(\tilde h + \delta\phi) = \mathcal{E}(\tilde h) + \delta \sum_{x\in V}\sum_{x\sim y} \underbrace{\big(\tilde h(x) - \tilde h(y)\big)\big(\phi(x)-\phi(y)\big)}_{(*)}$$

$$+ \delta^2 \sum_{x\in V}\sum_{x\sim y} \cdots$$

if $(*) > 0$, then take small negative δ. Then the third term is small and second term dominates.

~~If~~ Then $\mathcal{E}(\tilde{h} + \delta\phi)$ is made less than $\mathcal{E}(\tilde{h})$. $\lightning$

$$\Rightarrow \sum_{x \in V} \sum_{y \sim x} \left(\tilde{h}(x) - \tilde{h}(y)\right)\left(\phi(x) - \phi(y)\right) w_{xy} + \delta^2 \cdots = 0 \quad \forall \phi.$$

Let $\phi(\cdot) = \delta_x(\cdot)$, for ~~all~~ for some $x \neq a, z$.

$\overset{(???)}{\Longrightarrow} \tilde{h} = h$

② $\mathcal{E}(h)$

$$= \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \left(h(y) - h(x)\right)^2 w_{xy}$$

$$= -\frac{1}{2} \sum_{x \in V} \left(\sum_{y \sim x} \left(h(y) - h(x)\right) w_{xy} \right) h(x)$$

$$- \frac{1}{2} \sum_{y \in V} \left(\sum_{x \sim y} \left(h(x) - h(y)\right) w_{yx} \right) h(y)$$

$$= \sum_{y \sim z} w_{yz} \left[h(z) - h(y) \right]$$

$$= \|\theta\|$$

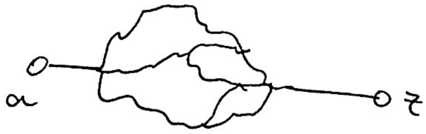
~~How to describe~~

With the above two propositions, we will be able to show the recurrence / transience of the network ~~Rw~~ RW on the random conductance model.

Lecture 5

We will now put the propositions together.

$$\left(R_{\text{eff}}(a \leftrightarrow z)\right)^{-1} = \inf_{\substack{h(a)=0 \\ h(z)=1}} \mathcal{E}(h),$$



$$\mathcal{E}(h) = \sum_{x\sim y} \omega_{xy}\left(h(y)-h(x)\right)^2$$

$$\mathbb{P}^{\omega}[\tau_z < \tau_a^+] = \frac{1}{\pi(a)\, R_{\text{eff}}(a \leftrightarrow z)}$$

$$\mathbb{P}^{\omega}[\tau_\infty < \tau_a^+] = \lim_{n\to\infty} \mathbb{P}^{\omega}[\tau_z^{(n)} < \tau_a^+]$$

$$= \lim_{n\to\infty} \frac{1}{\pi(a)\, R_{\text{eff}}(a \leftrightarrow z_n)}$$

$$R_{\text{eff}}(a \leftrightarrow \infty) := \lim_{n\to\infty} R_{\text{eff}}(a \leftrightarrow z_n)$$

Recall RCM: $\omega \in (\Lambda^{-1}, \Lambda)$, ~~$\Lambda \in (0,1$~~ $\Lambda \geq 1$

<u>Thm</u> RW on RCM is

Recurrent $d = 1, 2$

Transient $d \geq 3$

~~Euclidean energy~~ Dirichlet energy on Euclidean lattice

<u>Prf</u> Recall classical RW on $\mathbb{Z}^d$: $\inf_{\substack{h(a)=0 \\ h(z_n)=1}} \overline{\mathcal{E}}(h) \xrightarrow{n\to\infty} \begin{cases} 0 & d=1,2 \\ >0 & d\geq 3 \end{cases}$ (by Polya's theorem)

Notice that $\underbrace{\inf_{\substack{h(a)=0 \\ h(z_n)=1}} \overline{\mathcal{E}}(h)}_{\asymp}$

$$\inf_{\substack{h(a)=0 \\ h(z_n)=1}} \mathcal{E}(h)$$

$$\mathcal{E}(h) = \sum_{x \sim y} \omega_{xy} \left(h(y) - h(x)\right)^2$$

$$\leq \sum_{x \sim y} \Lambda \left(h(y) - h(x)\right)^2$$

$$\leq \Lambda \overline{\mathcal{E}}(h)$$

~~Other side is similar. Hence,~~

$$\cancel{\Lambda^{-1} \overline{\mathcal{E}}(h) \leq \inf_{\substack{h(a)=0 \\ h(z_n)=1}} \mathcal{E}(h) \leq \Lambda \overline{\mathcal{E}}(h)}$$

Expect: $\bar{h}$ is arg inf $\overline{\mathcal{E}}(\bar{h})$ h is arg inf $\mathcal{E}(h)$

Then $$\mathcal{E}(h) \leq \mathcal{E}(\bar{h}) \leq \Lambda \overline{\mathcal{E}}(\bar{h})$$

$$\inf \mathcal{E}(h) \leq \Lambda \inf \overline{\mathcal{E}}(\bar{h})$$

Other side is same. Hence,

$$\Lambda^{-1} \overline{\mathcal{E}}(h) \leq \inf_{\substack{h(a)=0 \\ h(z_n)=1}} \mathcal{E}(h) \leq \Lambda \overline{\mathcal{E}}(h).$$

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