Selected Notes on Schramm-Loewner Evolutions

Baining Zhang

April 29, 2025

Contents

1 Riemann Mapping Theorem

Let $\mathbb{D}$ denote the open unit disc at the origin. $\mathbb{H}$ is usually the upper half-plane $\{z\,;\, \operatorname{Im} z > 0\}$ unless otherwise specified. All 'morphisms' (e.g. automorphisms) are conformal.

1.1 Conformal radius

This section references [Bel20].

In the proof of the Riemann mapping theorem, we maximised $f'(z_0)$ to so that 'the function grows fast enough to fill up the whole disc'. Consider an alternative extremal formulation. Assume that Ω is a simply connected domain such that $\hat{\mathbb{C}} \setminus \Omega$ contains at least two points. We denote by $\mathscr{F}$ the family of all univalent maps on Ω with $f(z_0) = 0$ and $f'(z_0) = 1$. Consider the functional $f \mapsto \sup |f(z)|$. It is minimised by the unique (why?) univalent map onto the disc of radius

$$R = \min_{f \in \mathscr{F}} \sup_{z \in \mathbb{D}} |f(z)|.$$

This radius is called the conformal radius of the domain Ω at z_0 and is denoted by $\rho(z_0, \Omega)$.

We have an alternative definition of the conformal radius.

Proposition 1 (conformal radius)**.** Let Ω be a simply connected domain. Then there exists a unique univalent $F : \mathbb{D} \to \Omega$ with $F(0) = z_0$ such that

$$F'(0) = \rho(z_0, \Omega).$$

Proof. Take the Riemann map $\tilde{f} : \Omega \to \mathbb{D}$ univalent with $\tilde{f}(z_0) = 0$ and $\tilde{f}'(z_0) > 0$. Define $F = \tilde{f}^{-1}$. Then $F(0) = z_0$ and $F'(0) = 1/\tilde{f}'(z_0)$. Recall that $\tilde{f}$ is the function that maximises $\tilde{f}'(z_0)$ in

$$\left\{ \tilde{f} \text{ univalent on } \Omega\,;\, \operatorname{im} \tilde{f} \subseteq \mathbb{D}, \tilde{f}(z_0) = 0, \tilde{f}'(z_0) > 0 \right\}.$$

Hence, F is the function that minimises $F'(0)$. Consider the f in the previous discussion. We have $f = \tilde{f}/\tilde{f}'(z_0)$. Then $\operatorname{im} f = [1/\tilde{f}'(z_0)]\mathbb{D}$. So

$$R = \frac{1}{\tilde{f}'(z_0)} = F'(0).$$

Uniqueness of F follows from that of $\tilde{f}$ from the RMT. □

We don't necessarily need the RMT normalisation to get $F'(0) > 0$ to obtain the conformal radius. We can equivalently define it as simply

$$\rho(z_0, \Omega) = |F'(0)|$$

for any univalent $F : \mathbb{D} \to \Omega$ with $F(0) = z_0$. This is well-defined because such F are unique up to a precomposition of rotation of $\mathbb{D}$, and hence $G(z) = F(e^{i\theta} z)$ has derivative $G'(z) = e^{i\theta} F'(e^{i\theta} z)$ so their moduli are equal.

Proposition 2. Given a simply connected domain Ω. Then the univalent map $F : \mathbb{D} \to \Omega$ such that $F(0) = z_0$ for a fixed $z_0 \in \Omega$ is unique up to a rotation.

Proof. We consider $F_2^{-1} \circ F_1 \in \mathrm{Aut}(\mathbb{D})$. This is an automorphism of the unit disc that fixes 0. From the characterisation by Möbius transformations above, we see that $F_2^{-1} \circ F_1(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. $\square$

The conformal radius is strictly increasing: if $\Omega_1 \subseteq \Omega_2$, then $\rho(z_0, \Omega_1) \leq \rho(z_0, \Omega_2)$ where equality holds if and only if $\Omega_1 = \Omega_2$. This is given by the following proposition.

Proposition 3 (Lindelöf's principle)**.** Let $F_1 : \mathbb{D} \to \Omega_1$ and $F_2 : \mathbb{D} \to \Omega_2$ be univalent maps such that $F_1(0), F_2(0) = z_0$ and $\Omega_1 \subseteq \Omega_2$. Then $|F_1'(0)| \leq |F_2'(0)|$ where equality holds if and only if $F_2(z) = F_1(e^{i\theta} z)$ for some $\theta \in \mathbb{R}$.

Proof. By precomposing F_1 and F_2 with a rotation, we may assume that $F_1'(0), F_2'(0) > 0$. Define $f_1 = F_1^{-1}$ and $f_2 = F_2^{-1}$. Then f_1, f_2 are the unique Riemann maps. Consider $f_2|_{\Omega_1} \in \mathcal{F}$ that is, it is univalent on Ω_1 with $f_2(\Omega_1) \subseteq \mathbb{D}$, $f_2(z_0) = 0$ and $f_2(z_0) > 0$. Recall that f_1 is the maximiser of $f \in \mathcal{F} \mapsto f'(z_0)$, so we must have $f_1'(z_0) \geq f_2'(z_0)$ with equality holding if and only if $\Omega_1 = \Omega_2$. From this inequality, we derive the equivalent $F_1'(0) \leq F_2'(0)$.

The equality condition follows by thinking more about precomposing with rotation. $\square$

We will see the conformal radius again.

1.2 Other Normalisations

Thermodynamic Normalisation The following refers to to [Bel20] and [Kem17]. [Law05] also has a 'proof'. Sometimes, this is also called the hydrodynamic normalisation.

Definition 1 ($\mathbb{H}$-hull)**.** A compact subset $A \subseteq \mathbb{H} \cup \mathbb{R}$ is a $\mathbb{H}$-hull if $\mathbb{H} \setminus A$ is a simply connected domain and $\overline{A \cap \mathbb{H}} = A$.

Theorem 1. Let $\Omega^+ = \mathbb{H} \setminus A$, where A is a $\mathbb{H}$-hull. Then there is a unique univalent map $g_A : \Omega^+ \to \mathbb{H}$ such that the following holds:

$$\lim_{z \to \infty} (g_A(z) - z) = 0.$$

Equivalently, g_A has the following expansion at ∞:

$$g_A(z) = z + b_1 z^{-1} + b_2 z^{-2} + \dots .$$

Additionally, $b_1, b_2, \dots$ are real.

g_A are called mapping-out functions. As we will see later, the coefficient b_1 is the *half-plane capacity* of A.

Proof. Note that $\operatorname{im} g_A = \mathbb{H}$ and $g_A(\infty) = \infty$. RMT gives some univalent function $\tilde{g} : \Omega^+ \to \mathbb{D}$. We assume that $\tilde{g}$ can be continuously and injectively extended to $\mathbb{R} \cap \partial\Omega^+$. Then $\tilde{g}(\infty) \in \partial\mathbb{D}$.

We need to do something to $\tilde{g}$ to make it more like g_A. First, we define a Möbius transformation $\phi : \mathbb{D} \to \mathbb{H}$ given by

$$z \mapsto \frac{i - z}{i + z}.$$

Note that $\phi(-i) = \infty$. We define another Möbius transformation $\psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that is a rotation, given by

$$z \mapsto e^{i\left(-\frac{\pi}{2} - \arg(\tilde{g}(\infty))\right)} z.$$

Then we have $\psi(\tilde{g}(\infty)) = e^{i\left(-\frac{\pi}{2}\right)} = -i$. Let $g = \phi \circ \psi \circ \tilde{g} : \Omega^+ \to \mathbb{H}$.

$$\begin{tikzcd}
\infty \in \partial\Omega^+ \arrow[r, mapsto, "\tilde{g}"] \arrow[rrr, mapsto, bend right, "g"'] & \tilde{g}(\infty) \in \partial\mathbb{D} \arrow[r, mapsto, "\psi"] & -i \in \partial\mathbb{D} \arrow[r, mapsto, "\phi"] & \infty \in \partial\mathbb{H}
\end{tikzcd}$$

g is univalent since all the maps in the composition are univalent. With the continuous injective extension, and that $g(\infty) = \infty$, there exists some $B_{\text{chordal}}(\infty, r)$ such that $g(B_{\text{chordal}}(\infty, r) \cap \overline{\mathbb{R}}) \subseteq \overline{\mathbb{R}}$.

We want to expand g around ∞. But right now it is only defined on $B_{\text{chordal}}(\infty, r) \cap \mathbb{H}$. We extend it to all of $B_{\text{chordal}}(\infty, r)$ via the Schwarz reflection principle. Let $\Omega^- = \{\overline{z}\,;\, z \in \Omega^+\}$, $\Omega_0 = \partial\Omega^+ \cap \mathbb{R}$, and $\Omega = \Omega^+ \cup \Omega_0 \cup \Omega^-$. g is extended to be an analytic function $\hat{g}$ on Ω, given by

$$\hat{g}(z) = \begin{cases} g(z) & z \in \Omega^+ \cup \Omega_0, \\ \overline{g(\overline{z})} & z \in \Omega^-. \end{cases}$$

We see that $\hat{g}$ is still injective, given the injective extension to $\mathbb{R} \cap \partial\Omega$ and univalence of g in Ω^+.

Therefore, $\hat{g}$ has a simple pole at ∞, which means that its expansion at ∞ has the form

$$\hat{g}(z) = a_1 z + a_0 + a_{-1} z^{-1} + a_{-2} z^{-2} + \dots.$$

We want to show that $a_1 \in \mathbb{R}$. On $B_{\text{chordal}}(\infty, r) \cap \mathbb{R}$, $\hat{g}$ is real and we have the following:

$$\begin{aligned}
0 &= \lim_{R \in \mathbb{R} \to \infty} \left|\hat{g}(R) - \left(a_1 R + a_0 + a_{-1} R^{-1} + \dots\right)\right|, \\
&= \lim_{R \in \mathbb{R} \to \infty} |\hat{g}(R) - a_1 R|, \\
&= \lim_{R \in \mathbb{R} \to \infty} \sqrt{|\hat{g}(R) - \operatorname{Re} a_1 R|^2 + |\operatorname{Im} a_1 R|^2}, \\
&\geq \lim_{R \in \mathbb{R} \to \infty} |\operatorname{Im} a_1 R|.
\end{aligned}$$

Therefore, a_1 must be real. Furthermore, since $\hat{g}$ maps to the upper half-plane, $a_1 > 0$.

Now we consider $\frac{1}{a_1}\hat{g} : \Omega \to \mathbb{H}$, which is still univalent and real-valued for $r \in B_{\text{chordal}}(\infty, r) \cap \mathbb{R}$ with the following expansion at ∞:

$$\underbrace{\frac{1}{a_1}\hat{g}(r) - r}_{\in \mathbb{R}} = \frac{a_0}{a_1} + \frac{a_{-1}}{a_0} r^{-1} + \dots \quad \in \mathbb{R}$$

From an argument similar to above, $a_0 \in \mathbb{R}$. In the same manner, we can show that $a_j \in \mathbb{R}$ for all $j \leq 0$.

Finally, we define

$$g_A(z) = \frac{\hat{g}(z) - a_0}{a_1}$$

with the expansion at ∞

$$g_A(z) = z + b_1 z^{-1} + b_2 z^{-2} + \dots,$$

where $b_j \in \mathbb{R}$. It is obviously univalent. We check that g_A still maps to the upper half-plane. We actually have $g_A = \mu \circ \hat{g}$, where μ is the Möbius transformation given by

$$\mu(z) = \frac{z - a_0}{a_1},$$

which we observe to be an automorphism of the upper half-plane, by $a_1 > 0$.

Now we show uniqueness. Let g_1, g_2 both satisfy our conditions. Then $g_1 \circ g_2^{-1}$ is an automorphism of the upper half-plane. Furthermore, it fixes ∞. This implies

$$g_1 \circ g_2^{-1} = \alpha z + \beta \quad \iff \quad g_1(z) = g_2(\alpha z + \beta).$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$. We expand at ∞:

$$\lim_{z \to \infty} \left\{ \left[z + a_1 z^{-1} + a_2 z^{-2} + \dots \right] - \left[(\alpha z + \beta) + b_1(\alpha z + \beta)^{-1} + b_2(\alpha z + \beta)^{-2} + \dots \right] \right\} = 0,$$

where we obtain $\alpha = 1$, $\beta = 0$, and hence $g_1 \circ g_2^{-1} = \mathrm{id}_{\mathbb{H}}$, i.e. $g_1 = g_2$. $\square$

Unbounded domains

Theorem 2. Let $K \subseteq \mathbb{C}$ be a compact set such that $\Omega = \hat{\mathbb{C}} \setminus K$ is a simply connected domain. We have $\infty \in \Omega$. There exists a unique univalent $F_K : \mathbb{D}_- \to \Omega$ such that $F_K(\infty) = \infty$ and $F_K'(\infty) > 0$. Equivalently, the expansion of F_K at ∞ has the form

$$F_K(z) = a_1 z + a_0 + a_{-1} z^{-1} + \dots$$

where $a_1 > 0$.

Proof. We assume $0 \in K$. If not, let $z_0 \in K$, then $0 \in -z_0 + K$. We have $F_K(z) = F_{-z_0+K}(z - z_0)$, which satisfies our conditions if F_{-z_0+K} does. Hence, it suffices to prove the theorem for $K \ni 0$.

Let $\frac{1}{\Omega} = \left\{ \frac{1}{z} ;\ z \in \Omega \right\}$. Then $0 \in \frac{1}{\Omega}$. RMT gives $f_K : \mathbb{D} \to \frac{1}{\Omega}$ such that $f_K(0) = 0$ and $f_K'(0) > 0$. Let $F_K(z) = 1/f_K(1/z)$.

$$\begin{tikzcd} \mathbb{D}_- \arrow[r, "\frac{1}{z}"] \arrow[rrr, bend right, "F_K"'] & \mathbb{D} \arrow[r, "f_K"] & \frac{1}{\Omega} \arrow[r, "\frac{1}{z}"] & \Omega \end{tikzcd}$$

F_K is a composition of univalent functions, so it is univalent. We will now show that this satisfies our normalisation conditions.

By injectivity, f_K has a simple zero at 0. So its expansion at 0 has the form

$$f_K(z) = z \left(f_K'(0) + b_1 z + b_2 z^2 + \dots \right).$$

Hence,

$$F_K(z) = \frac{z}{f_K'(0)} + a_0 + a_{-1} z^{-1} + \dots.$$

From this, we obtain $F_K(\infty) = \infty$, and $F_K'(\infty) = \frac{1}{f_K'(0)} > 0$.

The uniqueness of F_K comes directly from the uniqueness of f_K given by the RMT. $\square$

2 Theory of Univalent Functions

2.1 Bieberbach-Koebe Theory

Definition 2.

$$\begin{aligned}
\mathcal{S} &= \{f \text{ univalent on } \mathbb{D}\,;\ f(0)=0, f'(0)=1\}\,.\\
\mathcal{S}^* &= \{f \text{ univalent on } \mathbb{D}\,;\ f(0)=0, f'(0)>0\}\,.\\
\Sigma &= \left\{g \text{ univalent on } \mathbb{D}_-\,;\ g(z)=z+b_0+b_1z^{-1}+\dots\right\}.\\
\Sigma' &= \{g\in\Sigma\,;\ 0\in\mathbb{C}\setminus g(\mathbb{D}_-)\}\,.\\
\mathcal{A} &= \{\Omega\subseteq\mathbb{C}\,;\ \Omega \text{ is simply connected and } \Omega\neq\mathbb{C},\emptyset\}\,.
\end{aligned}$$

Alternatively,

$$\mathcal{S}=\left\{f \text{ univalent on } \mathbb{D}\,;\ f(z)=z+a_2z^2+a_3z^3+\dots\right\}.$$

Definition 3. We define the Koebe function $f_{\text{Koebe}}:\mathbb{D}\to\mathbb{C}$, given by

$$f_{\text{Koebe}}(z)=z+2z^2+3z^3+\dots=\sum_{n=1}^{\infty}nz^n.$$

Proof. We need to show that this series converges in $\mathbb{D}$. But this is easy

$$\frac{1}{\limsup\sqrt[n]{n}}=1.$$

By absolute convergence, we have

$$f_{\text{Koebe}}(z)=\frac{z}{(1-z)^2}\quad(\text{in } \mathbb{D}).$$

□

This function is in $\mathcal{S}$, but it is difficult to determine this from the definition.

Proposition 4. $f_{\text{Koebe}}\in\mathcal{S}$.

Proof. We can rewrite it as

$$f_{\text{Koebe}}(z)=\frac{1}{4}\left(\frac{1+z}{1-z}\right)^2-\frac{1}{4}.$$

$$\mathbb{D}\xrightarrow{\frac{1+z}{1-z}}\mathbb{H}_{\text{right}}\xrightarrow{z^2}\mathbb{C}\setminus(-\infty,0]\xrightarrow{\frac{1}{4}z-\frac{1}{4}}\mathbb{C}\setminus(-\infty,\tfrac{1}{4}]$$

We see that it is a composition of univalent functions, so it is univalent. □

This will be an important example which demonstrates that many of the theorems which we are going to prove are indeed the best possible results.

We will present two theorems that relate geometric properties of univalent functions to their analytic properties.

Theorem 3 (Gronwall's area theorem)**.** Let $g(z) = z + \sum_{n=1}^{\infty} b_n z^{-n} \in \Sigma'$. Then $g(\mathbb{D}_-) = \mathbb{C} \setminus K$ for some compact set K. Its area is given by

$$\lambda_2(K) = \pi \left(1 - \sum_{n=1}^{\infty} n |b_n|^2 \right).$$

Proof. If g can be extended continuously to $\partial\mathbb{D}$, then we would directly apply Green's theorem to $\partial\mathbb{D}$ to compute the area. But this is might not be the case, and this problem is easy to bypass anyway.

Let $r > 1$. We define K_r to be the region enclosed by $g(r\partial\mathbb{D})$. As $r \downarrow 1$, we have $\lambda_2(K_r) \downarrow \lambda_2(K)$, which will give us the desired result.

We apply Green's theorem

$$\begin{aligned} \lambda_2(K_r) &= \frac{1}{2i} \int_{r\partial\mathbb{D}} \overline{w} \, \mathrm{d}w = \frac{1}{2i} \int_{r\partial\mathbb{D}} \overline{g(z)} g'(z) \, \mathrm{d}z \,, \\ &= \frac{1}{2i} \int_{r\partial\mathbb{D}} \left(\overline{z} + \sum_{n=0}^{\infty} \overline{b_n} \overline{z}^{-n} \right) \left(1 - \sum_{n=1}^{\infty} n b_n z^{-n-1} \right) \mathrm{d}z \,. \end{aligned}$$

Since this is quite tricky, we will do it here. Hopefully the argument is correct. Notice that on $r\partial\mathbb{D}$, $\overline{z} = \frac{r^2}{z}$. We breakdown the sum.

$$\lambda_2(K_r) = \frac{1}{2i} \int_{r\partial\mathbb{D}} \left(\overline{z} - \frac{r^2}{z} \sum_{n=1}^{\infty} n b_n z^{-n-1} + \sum_{n=0}^{\infty} \overline{b_n} \frac{z^n}{r^{2n}} - \left(\sum_{n=0}^{\infty} \overline{b_n} \frac{z^n}{r^{2n}} \right) \left(\sum_{n=1}^{\infty} n b_n z^{-n-1} \right) \right) \mathrm{d}z \,.$$

$$\frac{1}{2i} \int_{r\partial\mathbb{D}} \overline{z} \, \mathrm{d}z = \pi r^2,$$

by Green's theorem.

We move on to the next part.

$$\frac{1}{2i} \int_{r\partial\mathbb{D}} \left(r^2 \sum_{n=1}^{\infty} n b_n z^{-n-2} \right) \mathrm{d}z = 0$$

The series converges on $\left\{ z \,;\, |z| \geq \frac{r+1}{2} \right\}$, which contains $r\partial\mathbb{D}$. Its terms have degree ≤ -3 and hence the individual terms have primitives. So for any $N \in \mathbb{N}$,

$$\int_{r\partial\mathbb{D}} \left(\sum_{n=1}^{N} n b_n z^{-n-2} \right) = 0,$$

and

$$\left| \int_{r\partial\mathbb{D}} \left(\sum_{n=N+1}^{\infty} n b_n z^{-n-2} \right) \mathrm{d}z \right| \leq \sum_{n=N+1}^{\infty} \frac{n|b_n|}{r^{-n-2}} \to 0 \quad \text{as} \quad N \to \infty.$$

The third term is analytic. We can show that it converges on $r^2\mathbb{D} \supsetneq r\mathbb{D}$ by recalling that

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|b_n|}} \geq 1,$$

which gives us

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{|b_n| \cdot r^{-2n}}} \geq r^2.$$

Hence, the series is analytic on a simply connected domain $r^2\mathbb{D}$ containing $r\partial\mathbb{D}$. We apply Cauchy's theorem to get

$$\frac{1}{2i}\int_{r\partial\mathbb{D}} \sum_{n=0}^{\infty} \overline{b_n}\frac{z^n}{r^{2n}}\,\mathrm{d}z = 0.$$

Now for the final term, we note that in the product, only $z^n \cdot z^{-n-1}$ can produce a nonzero integral.

$$\begin{aligned}
&\frac{1}{2i}\int_{r\partial\mathbb{D}} \left(\sum_{n=0}^{\infty} \overline{b_n}\frac{z^n}{r^{2n}}\right)\left(\sum_{n=1}^{\infty} nb_n z^{-n-1}\right)\mathrm{d}z\\
=&\frac{1}{2i}\int_{r\partial\mathbb{D}} \left(\cdots + c_1 z + c_0 + \left(\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right) z^{-1} + c_{-2}z^{-2} + \ldots\right),\\
=&\frac{1}{2i}2\pi i\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n},\\
=&\pi\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}.
\end{aligned}$$

Of course, we need to show that everything we did is legitimate.

$$\sum_{n=0}^{\infty} \overline{b_n}\frac{z^n}{r^{2n}}$$

is absolutely convergent on $r^2\mathbb{D}$.

$$\sum_{n=1}^{\infty} nb_n z^{-n-1}\,\mathrm{d}z$$

is absolutely convergent on $\mathbb{D}_-$. So the series expansion in the second line is absolutely convergent on the open annulus $\left\{z\,;\,1 < |z| < r^2\right\}$. We split it into three integrals.

Even though the whole series only converges on the annulus, the nonnegative power part is still absolutely convergent on all of $r^2\mathbb{D}$, so we again apply Cauchy's theorem.

$$\int_{r\partial\mathbb{D}} \sum_{n=0}^{\infty} c_n z^n\,\mathrm{d}z = 0$$

The z^{-1} term is straightforward.

$$\int_{r\partial\mathbb{D}} \left(\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right) z^{-1}\,\mathrm{d}z = 2\pi i\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}.$$

We argue as before to show that

$$\int_{r\partial\mathbb{D}} \sum_{n=2}^{\infty} c_{-n} z^{-n} = 0.$$

Finally, we combine the above and obtain

$$\lambda_2(K_r) = \pi \left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \right).$$

We let $r \downarrow 1$ and obtain the desired formula. $\square$

Lemma 1. If $f \in \mathcal{S}$, then there is an odd function $h \in \mathcal{S}$ such that

$$[h(z)]^2 = f(z^2) \quad z \in \mathbb{D}.$$

Proof. Recall that f is injective, so it has a simple zero around 0 and $z \mapsto \frac{1}{z}f(z)$ is analytic and nonzero on all of $\mathbb{D}$. Since $\mathbb{D}$ is a simply connected domain, this transformation has a square root g, an analytic function, such that $[g(z)]^2 = \frac{1}{z}f(z)$. We define $h(z) = zg(z^2)$. We see that this is analytic and odd, with $[h(z)]^2 = f(z^2)$, $h(0) = 0$, and $h'(0) = 1$. If $h(z_1) = h(z_2)$, $f(z_1^2) = f(z_2^2)$, so $z_1 = \pm z_2$. Hence, $h(z_2) = h(z_1) = \pm h(z_2)$. So $\pm$ must be $+$ and $z_1 = z_2$. So h is univalent on $\mathbb{D}$, i.e. $h \in \mathcal{S}$. $\square$

Proposition 5 (Bieberbach)**.** If $f \in \mathcal{S}$, then $|a_2| \leq 2$. $|a_2| = 2$ if and only if $f = e^{i\theta} f_{\text{Koebe}}$.

Proof. f has a series expansion on $\mathbb{D}$

$$f(z) = z + a_2 z^2 + \dots.$$

For f, we have h provided by the above lemma. It also has a series expansion on $\mathbb{D}$

$$h(z) = z + b_2 z^2 + \dots.$$

Recalling that $[h(z)]^2 = f(z^2)$, we have

$$z^2 + a_2 z^4 + \dots = \left(z + b_2 z^2 + b_3 z^3 \dots\right)^2 = z^2 + 2b_2 z^3 + (2b_3 + b_2)z^4 + \dots$$

We see that $b_2 = 0, b_3 = \frac{a_2}{2}$. So

$$h(z) = z\left(1 + \frac{a_2}{2}z^2 + \dots\right) \quad \text{in} \quad \mathbb{D}.$$

Consider the function

$$g(z) = \frac{1}{h(\frac{1}{z})} = \frac{z}{1 + \frac{a_2}{2}z^{-2} + \dots} = z - \frac{a_2}{2}z^{-1} + \dots \in \Sigma' \quad \text{on} \quad \mathbb{D}_-.$$

A corollary of the area theorem implies

$$|a_2| \leq 2.$$

It is easy to see that $f = e^{i\theta} f_{\text{Koebe}}$ implies $|a_2| = 2$. For the other direction, suppose $|a_2| = 2$, then for g defined above, $\sum n|b_n| = \frac{|a_2|}{2} = 1$, which means that $\frac{a_2}{2} = e^{i\theta}$ for some $\theta \in \mathbb{R}$. So

$$g(z) = z - e^{i\theta} z^{-1}$$

by a corollary of the area theorem. From this, we also get

$$h(z) = \frac{z}{1 - e^{i\theta} z^2} \quad \implies \quad f(z^2) = \frac{z^2}{(1 - e^{i\theta} z^2)^2}.$$

There is no weird square root thing going on here, we directly obtain

$$f(z) = \frac{z}{(1 - e^{i\theta} z)^2} \quad \text{for} \quad z \in \mathbb{D}.$$

So $f(z) = e^{-i\theta} f_{\text{Koebe}}(e^{i\theta} z)$ for some $\theta \in \mathbb{R}$. $\square$

In fact, we have an extension of this theorem to all n.

Theorem 4 (de Branges' Theorem, formerly Bieberbach's Conjecture)**.** If $f \in \mathcal{S}$, then $|a_n| \leq n$.

Notice how the rotations of the Koebe function show that the inequality is sharp. We will continue to see this.

Theorem 5 (Koebe 1/4)**.** If $f \in \mathcal{S}$, then $f(\mathbb{D}) \supseteq \frac{1}{4}\mathbb{D}$. If there exists some $w \in \frac{1}{4}\partial\mathbb{D}$ but $w \notin f(\mathbb{D})$, then f is a rotation of the Koebe function.

Proof. We will show the contrapositive. Let $w \notin f(\mathbb{D})$. We want to show that $w \not\subseteq \frac{1}{4}\mathbb{D}$.

We define

$$\phi(z) = \frac{w f(z)}{w - f(z)}.$$

This is the composition of a Möbius transformation with f, hence ϕ is univalent. Furthermore, we have for $z \in \mathbb{D}$

$$\phi(z) = \frac{w f(z)}{w - f(z)} = \frac{wz + w a_2 z^2 + \dots}{w - z - a_2 z^2 - \dots} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \dots \in \mathcal{S}.$$

Bieberbach's theorem implies

$$\left| a_2 + \frac{1}{w} \right| \leq 2 \quad \text{and} \quad |a_2| \leq 2.$$

Hence,

$$\left| \frac{1}{w} \right| \leq |a_2| + \left| a_2 + \frac{1}{w} \right| \geq 4,$$

which gives us the desired conclusion.

Suppose $|w| = \frac{1}{4}$. Then

$$2 \geq |a_2| \geq \left| \frac{1}{w} \right| - \left| a_2 + \frac{1}{w} \right| \geq 2.$$

So $|a_2| = 2$. Bieberbach's theorem implies that f is the Koebe function. $\square$

Corollary 1. Let $f : \mathbb{D} \to \Omega$ be a function from $\mathcal{S}$. Then $d(0, \partial\Omega) \in [1/4, 1]$.

Proof. We first show $d(0, \partial\Omega) \le 1$. Suppose $d(0, \partial\Omega) = 1 + \varepsilon$ for some $\varepsilon \ge 0$. Then $\mathbb{D} \subseteq f(\Omega)$. Consider $f^{-1}|_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$, where $f^{-1}(0) = 0$ and $(f^{-1})'(0) = 1$ since $f \in \mathcal{S}$. By Schwarz's lemma, $f^{-1}(z) = z$ for all $z \in \mathbb{D}$. The Identity Theorem implies that $f^{-1}(z) = z$ for all $z \in \Omega$. Hence, $f(z) = z$. Then $\Omega = \mathbb{D}$ and $d(0, \partial\Omega) = 1$. So $\varepsilon = 0$ and $d(0, \partial\Omega) \le 1$.

Since $\frac{1}{4}\mathbb{D} \subseteq f(\mathbb{D}) = \Omega$, $d(0, \partial\Omega) \ge \frac{1}{4}$. □

We can generalise the Koebe $1/4$ theorem to all $f \in \mathcal{S}^*$. Through the bijection with simply connected domains, this gives us a geometric result which relates the in-radius $d(z_0, \partial\Omega)$ to the conformal radius $\rho(z_0, \Omega)$.

Corollary 2. Let $\Omega \in \mathcal{A}$ be a simply connected domain which is not $\emptyset$ or $\mathbb{C}$, and $z_0 \in \Omega$. Then

$$1 \le \frac{\rho(z_0, \Omega)}{d(z_0, \partial\Omega)} \le 4.$$

Proof. By translation, we can assume that $z_0 = 0$. Let $\tilde{f} : \mathbb{D} \to \Omega \in \mathcal{S}^*$ be the univalent function corresponding to $\Omega \in \mathcal{A}$. Then

$$f(z) = \frac{\tilde{f}(z)}{f'(0)} \in \mathcal{S}.$$

Recall that $f'(0) = \rho(z_0, \Omega)$. We apply the previous corollary to F to get

$$\begin{aligned}
\frac{1}{4} &\le d(0, \partial[\tilde{f}(\mathbb{D})]) \le 1, \\
\frac{1}{4} &\le \frac{d(0, \partial\Omega)}{f'(0)} \le 1, \\
4 &\ge \frac{f'(0)}{d(0, \partial\Omega)} \ge 1, \\
1 &\le \frac{\rho(0, \Omega)}{d(0, \partial\Omega)} \le 4.
\end{aligned}$$

□

Theorem 6 (Koebe distortion)**.** Let $f : \Omega \to \Omega'$ be a univalent map and $z \in \Omega$. Then let $w = f(z)$ and

$$\frac{1}{4} d\left(w, \partial\Omega'\right) \le |f'(z)| d(z, \partial\Omega) \le 4 d\left(w, \partial\Omega'\right).$$

Proof. We decompose f into $h \circ g^{-1}$ given by

$$\begin{tikzcd}
\Omega \arrow[r, "g^{-1}"'] \arrow[rr, bend left, "f"] & \mathbb{D} \arrow[r, "h"'] & \Omega'
\end{tikzcd}$$

with no particular normalisation. Let $\zeta = h^{-1}(w) = g^{-1}(z)$.

$$f'(z) = h'(g^{-1}(z))(g^{-1})'(z) = \frac{h'(\zeta)}{g'(\zeta)}.$$

We apply the previous corollary and obtain

$$1 \leq \frac{|h'(\zeta)|}{d(w, \partial\Omega')} \leq 4,$$
$$1 \geq \frac{d(z, \partial\Omega)}{|g'(\zeta)|} \geq \frac{1}{4}.$$

We multiply these two inequalities together and obtain

$$\frac{1}{4} \leq \frac{d(z, \partial\Omega)}{d(w, \partial\Omega')}|f'(z)| \leq 4$$

which leads to the desired result. □

2.2 Capacity and Half-Plane Capacity

This section refers mostly to [Law05].

2.2.1 Capacity

Definition 4. A (compact) hull K is a compact connected subset of $\mathbb{C}$ larger than a single point such that $\mathbb{C} \setminus K$ is connected.

Recall the 'Unbounded domains' normalisation for the RMT. We have shown that any compact hull K has a unique univalent mapping $F_K : \mathbb{D}_- \to \hat{\mathbb{C}} \setminus K$ with the expansion at ∞

$$F_K(z) = a_1 z + a_0 + \sum_{n=1}^{\infty} a_{-n} z^{-n}.$$

For compact hulls, we have a new notion of size given by this series expansion:

Definition 5. The (logarithmic) capacity $\operatorname{cap}(K)$ is given by

$$\operatorname{cap}(K) = \log a_1 = \log \left[\lim_{z\to\infty} \frac{F_K(z)}{z} \right].$$

By inspecting the expansion of F_K, we see that for $w \in \mathbb{C}$, and $a > 0$, we have

$$F_{K+w} = F_K \quad \text{and} \quad F_{aK} = aF_K.$$

Hence,

$$\operatorname{cap}(K + w) = \operatorname{cap}(K) \quad \text{and} \quad \operatorname{cap}(aK) = \operatorname{cap}(A) + \log a.$$

We follow the notation of [Law05]. Let $\mathcal{H}^*$ denote the set of hulls and $\mathcal{H}$ those containing the origin. Let $\mathcal{H}_0^*$ and $\mathcal{H}_0$ be sets of hulls in $\mathcal{H}^*$ and $\mathcal{H}$ with zero capacity.

Let $g_K = F_K^{-1}$, $f_K(z) = \frac{1}{F_K(1/z)}$, so

$$f_K'(0) = e^{-\operatorname{cap}(K)}.$$

Hence, we see that $f_K \in \mathcal{S}$ if and only if $K \in \mathcal{H}_0$. For any hull K, let $\operatorname{rad}(K) = \sup\{|z|\,;\, z \in K\}$.

Capacity is strictly monotonic.

Proposition 6. If $K_1, K_2 \in \mathcal{H}$, with $K_1 \subseteq K_2$, then $\operatorname{cap}(K_1) \leq \operatorname{cap}(K_2)$, where equality holds if and only if $K_1 = K_2$.

Proof. Consider f_{K_1} and f_{K_2}. Then $f_{K_1}^{-1} \circ f_{K_2} \in \operatorname{Aut}(\mathbb{D})$ and $(f_{K_1}^{-1} \circ f_{K_2})(0) = 0$. By Schwarz's lemma $\left|(f_{K_1}^{-1} \circ f_{K_2})'(0)\right| \leq 1$, which gives $f_{K_2}'(0) \leq f_{K_1}'(0)$ and hence $\operatorname{cap}(K_1) \leq \operatorname{cap}(K_2)$.

If $\operatorname{cap}(K_1) = \operatorname{cap}(K_2)$, $f_{K_2}'(0) = f_{K_1}'(0)$. Then $\left|(f_{K_1}^{-1} \circ f_{K_2})'(0)\right| = 1$. By Schwarz's lemma, $(f_{K_1}^{-1} \circ f_{K_2})(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. Taking derivative on both sides, we have

$$\frac{f_{K_2}'(0)}{f_{K_1}'(0)} = e^{i\theta}.$$

But recall that $f_{K_1}'(0), f_{K_2}'(0) > 0$, so $e^{i\theta} > 0$. So $e^{i\theta} = 1$. Finally, we have $(f_{K_1}^{-1} \circ f_{K_2})(z) = z$, i.e. $f_{K_1} = f_{K_2}$. ☐

Example 1. For $K = [-1, 1]$,

$$F_K(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Proposition 7. If $K \in \mathcal{H}_0$, then $1 \leq \operatorname{rad}(K) \leq 4$. Also, $[-4, 0] \in \mathcal{H}_0$.

2.2.2 Half-plane capacity

Let $\mathcal{Q}$ denote the set of compact $\mathbb{H}$-hulls.

Definition 6. If A is a $\mathbb{H}$-hull, the half-plane capacity, $\operatorname{hcap}(A)$, is defined by

$$\operatorname{hcap}(A) = \lim_{z \to \infty} z\left[g_A(z) - z\right].$$

Alternatively, we expand g_A around ∞

$$g_A(z) = z + \frac{\operatorname{hcap}(A)}{z} + O\left(\frac{1}{|z|^2}\right).$$

Similar to the case of compact hulls, if $r > 0$, $x \in \mathbb{R}$, and A is a $\mathbb{H}$-hull, then

$$g_{rA}(z) = r g_A\left(\frac{z}{r}\right) \quad \text{and} \quad g_{A+x}(z) = g_A(z - x) + x.$$

Hence,

$$\operatorname{hcap}(rA) = r^2 \operatorname{hcap}(A) \quad \text{and} \quad \operatorname{hcap}(A + x) = \operatorname{hcap}(A).$$

We want to consider the additive properties of the half-plane capacity. Let A, B be disjoint $\mathbb{H}$-hulls. We consider their mapping-out functions. Let's push A into $\mathbb{R}$ first. Consider $g_A(\mathbb{H} \setminus (A \cup B)) = g_A((\mathbb{H} \setminus A) \setminus B) = \mathbb{H} \setminus g_A(B)$ by injectivity. Now, we push the image of B under g_A into $\mathbb{R}$ by $g_{g_A(B)}$. Therefore, we see that $g_{A \cup B} = g_{g_A(B)} \circ g_A$. Now, we expand this around ∞ to look into the coefficients and obtain the following relation:

$$\operatorname{hcap}(A \cup B) = \operatorname{hcap}(A) + \operatorname{hcap}(g_A(B)).$$

Example 2. For $A = \overline{\mathbb{D}} \cap \mathbb{H}$,

$$g_A(z) = z + \frac{1}{z}.$$

$\mathrm{hcap}(\overline{\mathbb{D}}) = 1$

For $A' = (0, i]$, then

$$g_{A'}(z) = \sqrt{z^2 + 1} = z + \frac{1}{2z} + \dots.$$

$\mathrm{hcap}((0, i]) = \frac{1}{2}$.

Proposition 8. $\mathrm{hcap}(A)$ is non-negative and zero if and only if A is empty.

Proof. This is a restatement of Lemma 4.2 in [Kem17] in a more geometric form. The original statement is

The coefficient b_1 is nonnegative and $b_1 = 0$ only if g_A is the identity map.

Since we have defined $\mathrm{hcap}(A) = b_1$, the first part is obvious. If g_A is the identity map, then it is a conformal automorphism of $\mathbb{H}$. But recall that it is supposed to be a conformal isomorphism of $\mathbb{H} \setminus A$ and $\mathbb{H}$, so we can conclude that $A = \emptyset$. $\square$

3 Loewner Evolutions

3.1 Local Growth and Loewner Chains

This theorem gives all necessary and sufficient condition to the existence of a continuous driving term for g_t. The condition is called *local growth*.

Theorem 7. Let $(K_t)_{t\in[0,T]}$ be a growing family of hulls and g_t the associated conformal maps. Then following are equivalent:

- For all $t \in [0, T]$, $\text{hcap}(K_t) = 2t$ and for any $\varepsilon > 0$ there is $\delta > 0$ such that for each $t \in [0, T - \delta]$, there exists a bounded connected set $C \subseteq \mathbb{H} \setminus K$ with $\text{diam}(C) < \varepsilon$ such that C separates $K_{t+\delta} \setminus K_t$ from infinity in $\mathbb{H} \setminus K_t$.

- There is a continuous $W(t)$, $t \in [0, T]$ such that g_t is the solution of the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}.$$

Definition 7. A Loewner chain is the solution g_t of the Loewner equation with a continuous driving term.

4 Brownian Motion

4.1 First Properties

Proposition 9 (scaling invariance, Lemma 1.7, [Mör+10]). Suppose $(B(t))_{t\geq 0}$ is a standard Brownian motion and let $a > 0$. Then the process $(X(t))_{t\geq 0}$ defined by $X(t) = \frac{1}{a}B(a^2 t)$ is also a standard Brownian motion.

Proof. Continuity of paths, independence and stationarity of increments are obviously unchanged.

Recall that $B(t)$ are normally distributed. Observe that $X(t+h) - X(t) = \frac{1}{a}(B(a^2(t+h)) - B(a^2 t))$ is normally distributed with mean 0 and variance $\frac{1}{a^2}(a^2(t+h) - a^2 t) = h$. $\square$

4.2 Markov Property

This section refers entirely to [Mör+10]. Maybe you should go read [Mör+10] instead.

Theorem 8 (Markov property, Theorem 2.3, [Mör+10]). Suppose that $(B(t))_{t\geq 0}$ is a Brownian motion started in $x \in \mathbb{R}^d$. Let $s > 0$, then the process $(B(t+s) - B(s))_{t\geq 0}$ is again a Brownian motion started in the origin and it is independent of the process $(B(t))_{0\leq t\leq s}$.

Proof. It is easy to see that $(B(t+s) - B(s))_{t\geq 0}$ has the distribution of a standard Brownian motion.

The independence follows from the independence of the increments of the original Brownian motion $(B_t)_{t\geq 0}$ from $\mathcal{F}_s$. $\square$

If we have a Brownian motion $(B(t))_{t\geq 0}$, we have the natural filtration $(\mathcal{F}^0(t))_{t\geq 0}$. We see that the process $(B(t+s) - B(s))_{t\geq 0}$ is independent of the σ-algebra $\mathcal{F}^0(s)$. But actually, we can show that it is independent of a slightly larger σ-algebra $\mathcal{F}^+(s)$, given by

$$\mathcal{F}^+(s) = \bigcap_{t>s} \mathcal{F}^0(t).$$

This can be understood as having a infinitesimal glimpse into the future. Since it is a larger filtration, Brownian motion is also adapted to it. Surprisingly, the Markov property also holds for this filtration.

Theorem 9 (Theorem 2.5, [Mör+10]). For any $s \geq 0$, the process $(B(t+s) - B(s))_{t\geq 0}$ is independent of the σ-algebra $\mathcal{F}^+(s)$.

Proof. Let $s_n \searrow s$,. Then $\mathcal{F}^+(s) = \cap \mathcal{F}^0(s_n)$. By continuity,

$$B(t+s) - B(s) = \lim_{n\to\infty} [B(t+s_n) - B(s_n)].$$

Each of the $B(t+s_n) - B(s_n)$ is independent of $\mathcal{F}^0(s_n)$. Because s_n is strictly decreasing, the limit is independent of $\mathcal{F}^0(s_n)$ for all $n \in \mathbb{N}$ because you can just keep ignoring the first $n-1$ terms of the sequence. Hence, $B(t+s) - B(s)$ is independent of $\mathcal{F}^+(s)$. $\square$

This is a strictly larger σ-algebra. We shall illustrate this by considering the class of stopping times defined on each filtration.

Let F be closed in $\mathbb{R}^d$. Then $\tilde{H}(F) = \inf\{t \geq 0\,;\, B(t) \in F\}$, the hitting time of F is a stopping time with respect to $(\mathcal{F}_t^0)$ by the following:

$$\left\{\tilde{H}(F) \leq t\right\} = \bigcap_{n=1}^{\infty} \bigcup_{s \in \mathbb{Q} \cap (0,t)} \bigcap_{x \in \mathbb{Q}^d \cap F} \left\{B(s) \in B(x, \frac{1}{n})\right\} \in \mathcal{F}^0(t).$$

If $G \subseteq \mathbb{R}^d$ is open, then $\tilde{H}(G)$ is a stopping time with respect to $(\mathcal{F}^+(t))_{t\geq 0}$ but not necessarily with respect to $(\mathcal{F}^0(t))_{t\geq 0}$. We shall see this. By the continuity of BM, we have

$$\left\{\tilde{H}(G) \leq t\right\} = \bigcap_{s>t} \left\{\tilde{H}(G) < s\right\} = \bigcap_{s>t} \underbrace{\bigcap_{r \in \mathbb{Q}\cap(0,s)} \{B(r) \in G\}}_{\in \mathcal{F}^0(s)} \in \mathcal{F}^+(t).$$

We now explain why it is not necessarily a stopping time with respect to $(\mathcal{F}^0(t))_{t\geq 0}$. Suppose G is bounded and the starting point of the BM is not in $\overline{G}$. We fix the start of a sample path $\gamma : [0, t] \to \mathbb{R}^d$ with $\gamma(0, t) \cap \overline{G} = \emptyset$. If it goes into G immediately after $\gamma(t)$, then $\tilde{H}(G) = t$. If not, i.e. it sort of rebounds off the boundary, then $\tilde{H}(G) \neq t$. But this obviously cannot be decided by the information in $\mathcal{F}^0(t)$.

Of course, you can say that $\tilde{H}(G)$ isn't even the hitting time of G because $B(\tilde{H}(G)) \in \partial G$ but G is open so $B(\tilde{H}(G)) \notin G$. Why should we want this to be a stopping time? But the important thing is that you know that after $\tilde{H}(G)$, the BM will immediately go into G. And we might really want to study these kinds of times. This is where the infinitesimal glimpse into the future of $(\mathcal{F}^+(t))_{t\geq 0}$ is at work.

So we see that unlike the extended filtration, the natural filtration is not really compatible with the topology of $\mathbb{R}^d$. We can phrase this more precisely.

Proposition 10. The filtration $(\mathcal{F}_t^+)_{t\geq 0}$ is right-continuous, i.e.

$$\bigcap_{\varepsilon>0} \mathcal{F}^+(t+\varepsilon) = \mathcal{F}^+(t).$$

Proof. We note that

$$\bigcap_{\varepsilon>0} \mathcal{F}^+(t+\varepsilon) = \bigcap_{n=1}^{\infty} \mathcal{F}^+(t+\frac{1}{n}) = \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty} \mathcal{F}^0(t+\frac{1}{n}+\frac{1}{k}) = \mathcal{F}^+(t).$$

□

The following proposition illustrates the technical advantage of right-continuous filtrations.

Proposition 11. Let $(\mathcal{F}(t))_{t\geq 0}$ denote a right-continuous filtration. Suppose the random time T with values in $[0, +\infty]$ satisfies $\{T < t\} \in \mathcal{F}(t)$ for any $t \geq 0$, then T is a stopping time with respect to $(\mathcal{F}(t))_{t\geq 0}$.

Proof. We note that

$$\{T \leq t\} = \bigcap_{k=1}^{\infty} \left\{T < t + \frac{1}{k}\right\} \in \bigcap_{n=1}^{\infty} \mathcal{F}(t+\frac{1}{n}) = \mathcal{F}(t),$$

where the last step is by right continuity. □

For any stopping time T, we define the σ-algebra

$$\mathcal{F}^+(T) = \left\{A \in \mathscr{F}\,;\, A \cap \{T \le t\} \in \mathcal{F}^+(t) \forall t \ge 0\right\}.$$

This is just the collection of events that happen before T. In particular, the path of the BM before T, $\{B(t)\,;\, t \le T\}$ is $\mathcal{F}^+(T)$-measurable. And by right-continuity, we see that $\{T \le t\} = \{T < t\}$ for stopping times with respect to right-continuous filtrations.

We now prove the strong Markov property for stopping times.

Theorem 10 (strong Markov property, Theorem 2.16, [Mör+10])**.** For every a.s. finite stopping time T, the process

$$\{B(T+t) - B(T)\}$$

is a standard Brownian motion independent of $\mathcal{F}^+(T)$.

Proof. We first define a sequence of stopping times

$$T_n = (m+1)2^{-n} \quad \text{for some} \quad m \text{ such that } m \cdot 2^{-n} \le T < (m+1)2^{-n}$$

That is, we stop at the first time of the form $k2^{-n}$ after T. It is easy to see that $T_n \searrow T$ (recall that T is a.s. finite).

We write $B_k = (B_k(t))_{t\ge 0}$ for BM defined by

$$B\left(t + \frac{k}{2^n}\right) - B\left(\frac{k}{2^n}\right).$$

We also write $B_* = (B_*(t))_{t \ge 0}$ for

$$B_*(t) = B(t + T_n) - B(T_n).$$

First, we want to show that $B_*(t)$ is independent of $\mathcal{F}^{T_n}$ and then send n to infinity to retrieve our desired result.

Let $E \in \mathcal{F}^+(T_n)$. For every event $\{B_* \in A\}$, we have

$$\begin{aligned}
\mathbf{P}(\{B_* \in A\}) &= \sum_{k=0}^{\infty} \mathbf{P}\left(\{B_k \in A\} \cap E \cap \left\{T_n = k2^{-n}\right\}\right), \\
&= \sum_{k=0}^{\infty} \mathbf{P}\left(E \cap \left\{T_n = k2^{-n}\right\}\right).
\end{aligned}$$

In the first equality we are partitioning according to the value of T_n. The independence of $\{B_k \in A\}$ and $E \cap \{T_n = k2^{-n}\} \in \mathcal{F}^+(k2^{-n})$ is by the Markov property with respect to the filtration $\mathcal{F}^+$. Again by Markov property, we have $\mathbf{P}\{B_k \in A\} = \mathbf{P}\{B \in A\}$ by the Markov property. So we have

$$\begin{aligned}
&= \mathbf{P}\{B \in A\} \sum_{k=0}^{\infty} \mathbf{P}\left(E \cap \left\{T_n = k2^{-n}\right\}\right), \\
&= \mathbf{P}\{B \in A\}\, \mathbf{P}(E).
\end{aligned}$$

We have shown that B_* is independent of $\mathcal{F}^+(T_n)$.

In particular, we also retrieve $\mathbf{P}\{B_* \in A\} = \mathbf{P}\{B \in A\}$ by setting E to be the whole sample space. Hence we have also proved B_* is a BM.

Now we push this result to T. Since $T_n \searrow T$, we know that $\{B(s+T_n) - B(T_n)\}_{s \geq 0}$ is a BM independent of $\mathcal{F}^+(T)$. We consider the increments of $\{B(t+T) - B(T)\}$

$$B(s+t+T) - B(t+T) = \lim_{n \to \infty} B(s+t+T_n) - B(t+T_n).$$

The terms inside the limit are BM increments. Hence the limit is self are increments which are independent and normally distributed with mean zero and variance s. This process is also obviously a.s. continuous, so it is a BM. Moreover, all the increments and hence the process itself are independent of $\mathcal{F}^+(T)$. $\square$

5 Schramm-Loewner Evolutions

5.1 SLE as a Stochastic Loewner Chain

Let $\mathcal{K}$ denote the set of $\mathbb{H}$-hulls, and $\mathcal{L}$ denote the set of increasing families of $\mathbb{H}$-hulls of local growth, parametrised by the half-plane capacity.

Definition 8 (Definition 5.1, [Kem17]). Let $\kappa \geq 0$. A chordal SLE(κ) in $\mathbb{H}$ from 0 to ∞ is a stochastic Loewner chain with a driving process $(W_t)_{t\geq 0} = (\sqrt{\kappa}B(t))_{t\geq 0}$, where $(B(t))_{t\geq 0}$ is a standard one-dimensional Brownian motion. More explicitly, an SLE(κ) is a $\mathcal{L}$-valued process $(K_t)_{t\geq 0}$. We denote the measure associated to SLE(κ) by $\mu^{(\mathbb{H},0,\infty)}$.

Since g_t and K_t are equivalent. We will also call $(g_t)_{t\geq 0}$ an SLE(κ), but we also call it the Loewner flow.

We need to show that this process is actually well-defined.

Let's set up the probability space. We work on the canonical space of the Brownian motion $(B(t))_{t\geq 0}$. The sample space is $C[0, +\infty)$. The σ-algebra $\mathscr{F}$ is generated by the projections $(\pi_t(B))_{t\geq 0}$. We have the measure $\mathbb{P}$ of the law of $B(t)$.

In fact, $\mathscr{F}$ is a Borel σ-algebra of a certain topology:

Proposition 12 (Theorem 21.31, [Kle20]). $\mathscr{F} = \mathscr{B}(\tau)$, where τ is the compact-open topology on $C[0,\infty)$, i.e. topology of uniform convergence on compact subsets of $[0,\infty)$.

We now establish the continuity of the Loewner transform $L : C([0,\infty),\mathbb{R}), \to \mathcal{L}$, which maps driving processes to their corresponding Loewner chains.

Note that $\mathcal{L} \subseteq C([0,\infty),\mathcal{K})$, on which the topology is that of the compact-open topology induced by Carathéodory convergence on $\mathcal{K}$. Recall that we have characterisations of the Carathéodory convergence.

Proposition 13 (Theorem 3.19, [Bel20]). Let $\Omega_n = \mathbb{H} \setminus K_n$ and $g_n : \mathbb{H} \setminus K_n \to \mathbb{H}$. Then the domains Ω_n converges to $\Omega = \mathbb{H} \setminus K$ if and only if $g_n \to g = g_K$ uniformly on compact subsets of $\mathbb{H} \setminus K$.

Hence, it suffices to show the following.

Theorem 11 (Proposition 6.2, [Kem17]).
For any compact $G \subseteq \mathbb{H}$, there exists a constant C such that: If $(K_t^{(1)})_{t\geq 0}$ and $(K_t^{(2)})_{t\geq 0}$ are two Loewner chains such that $K_T^{(1)}$ and $K_T^{(2)}$ are subsets of $\overline{\mathbb{H}} \setminus G$, then

$$\|g_1 - g_2\|_{\infty,[0,T]\times G} \leq C \|W_1 - W_2\|_{\infty,[0,T]}.$$

Proof. According to [Kem17], this directly follows from the lemma below. $\square$

Lemma 2 (Lemma 6.2, [Kem17]). For any $\delta > 0$, $T > 0$, there exists $C(T,\delta)$ such that

$$|g_1(T,z_1) - g_2(T,z_2)| \leq C(T,\delta)(\|W_1 - W_2\|_{\infty,[0,T]} + |z_1 - z_2|)$$

for any $z_1, z_2 \in \mathbb{H}$ such that $\operatorname{Im} g_k(T,z_k) > \delta$ for $k = 1,2$, where g_1, g_2 are solutions to the Loewner equation with continuous driving terms $(W_k(t))_{t\in[0,T]}$, $k = 1,2$, resp.

Proof. Let $\psi(t) = g_1(t, z_1) - g_2(t, z_2)$. We want to bound $|\psi(t)|$.

We have

$$\begin{aligned}
\partial_t \psi(t) &= \frac{2}{g_1(t, z_1) - W_1(t)} - \frac{2}{g_2(t, z_2) - W_2(t)}, \\
&= \frac{-2[(g_2(t, z_2) - W_2(t)) - (g_1(t, z_1) - W_1(t))]}{(g_1(t, z_1) - W_1(t))(g_2(t, z_2) - W_2(t))}, \\
&= \zeta(t)(\psi(t) - D(t)),
\end{aligned}$$

where

$$\zeta(t) = \frac{-2}{(g_1(t, z_1) - W_1(t))(g_2(t, z_2) - W_2(t))} \quad \text{and} \quad D(t) = |W_1(t) - W_2(t)|.$$

We want solve for $\psi(t)$ in terms of $D(t)$ and $\zeta(t)$, then $|\psi(t)|$ will become easier to bound with $|D(t)|$. We write

$$\partial_t \left(e^{-\int_0^t \zeta(s) \mathrm{d}s} \psi(t) \right) = -\zeta(t) e^{-\int_0^t \zeta(s) \mathrm{d}s} D(t)$$

with the integrating factor. Hence, we have

$$\begin{aligned}
e^{-\int_0^t \zeta(s) \mathrm{d}s} \psi(t) - \psi(0) &= -\int_0^t \zeta(u) e^{-\int_0^u \zeta(s) \mathrm{d}s} D(u) \, \mathrm{d}u \,, \\
\psi(t) &= e^{\int_0^t \zeta(s) \mathrm{d}s} \psi(0) - \int_0^t \zeta(u) e^{\int_0^t \zeta(s) \mathrm{d}s - \int_0^u \zeta(s) \mathrm{d}s} D(u) \, \mathrm{d}u \,, \\
&= e^{\int_0^t \zeta(s) \mathrm{d}s} |z_1 - z_2| - \int_0^t \zeta(u) e^{\int_u^t \zeta(s) \mathrm{d}s} D(u) \, \mathrm{d}u \,.
\end{aligned}$$

We have

$$\begin{aligned}
\left| \int_0^t \zeta(u) e^{\int_u^t \zeta(s) \mathrm{d}s} D(u) \, \mathrm{d}u \right| &\leq \|D\|_{\infty, [0, T]} \int_0^t |\zeta(u)| e^{\int_u^t |\zeta(s)| \mathrm{d}s} \, \mathrm{d}u \,, \\
&= \|D\|_{\infty, [0, T]} \left(-e^{\int_u^t |\zeta|(s) \mathrm{d}s} \right) \Big|_0^t \,, \\
&= \|D\|_{\infty, [0, T]} \left(e^{\int_0^t |\zeta(s)| \mathrm{d}s} - 1 \right) .
\end{aligned}$$

It is quite clear now that our job is to bound $e^{\int_0^t |\zeta(s)| \mathrm{d}s}$, which will gives us the following:

$$|\psi(t)| \leq e^{\int_0^t |\zeta(s)| \mathrm{d}s} |z_1 - z_2| + \left(e^{\int_0^t |\zeta(s)| \mathrm{d}s} - 1 \right) \|D\|_{\infty, [0, T]} \,.$$

The Cauchy-Schwarz inequality gives

$$\int_0^t |\zeta(s)| \, \mathrm{d}s \leq \sqrt{I_1 I_2},$$

where

$$I_k = \int_0^t \frac{2 \, \mathrm{d}s}{|g_k(s, z_k) - W_k(s)|^2}.$$

Now, we will estimate I_k. Notice that

$$\partial_s \left(\log \frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(s, z_k)} \right) = \frac{\partial_s \left(\frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(s,z_k)} \right)}{\frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(s,z_k)}},$$

$$= \operatorname{Im} g_k(s, z_k) \cdot \frac{-1}{[\operatorname{Im} g_k(s, z_k)]^2} [\partial_s \operatorname{Im} g_k(s, z_k)],$$

$$= \frac{-1}{\operatorname{Im} g_k(s, z_k)} [\partial_s \operatorname{Im} g_k(s, z_k)].$$

From the Loewner equation, we have

$$\partial_s \operatorname{Re} g_k(s, z_k) + i\partial_s \operatorname{Im} g_k(s, z_k) = \frac{2(\overline{g_k(s, z_k)} - W_k(s))}{|g_k(s, z_k) - W_k(s)|^2},$$

$$\partial_s \operatorname{Im} g_k(s, z_k) = \frac{-2 \operatorname{Im} g_k(s, z_k)}{|g_k(s, z_k) - W_k(s)|^2}.$$

So

$$\partial_s \left(\log \frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(s, z_k)} \right) = \frac{2}{|g_k(s, z_k) - W_k(s)|}.$$

So we have

$$I_k = \int_0^t \partial_s \left(\log \frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(s, z_k)} \right) \mathrm{d}s = \log \frac{\operatorname{Im} z_k}{\operatorname{Im} g_k(s, z_k)}.$$

Now we want to approximate $\operatorname{Im} z_k$. This calculation is similar to that in the proof of Proposition 4.1, [Kem17]. Notice that $W_k(s)$ is always real, so the imaginary part of $g_k(s, z_k) - W_k(s)$ is the same as that of $g_k(s, z_k)$. Hence, we have

$$\partial_s \operatorname{Im} g_k(t, z_k) \geq \frac{-2 \operatorname{Im} g_n(s, z_n)}{|\operatorname{Im} g_k(s, z_k)|^2} = \frac{-2}{\operatorname{Im} g_k(s, z_k)}.$$

From this, with some creativity, we obtain

$$\partial_s \left[(\operatorname{Im} g_k(s, z_k))^2 \right] = 2 \operatorname{Im} g_k(s, z_k) (\partial_s \operatorname{Im} g_k(s, z_k)),$$

$$(\operatorname{Im} g_k(t, z_k))^2 - (\operatorname{Im} z_k)^2 \geq \int_0^t \frac{2 \operatorname{Im} g_k(s, z_k) \cdot (-2)}{\operatorname{Im} g_k(s, z_k)} \, \mathrm{d}t \,,$$

$$= -4t,$$

$$\operatorname{Im} z_k \leq \sqrt{\operatorname{Im} g_k(t, z_k)^2 + 4t}.$$

This gives us the appropriate bound

$$I_k \leq \log \frac{\sqrt{(\operatorname{Im} g_k(t, z_k))^2 + 4t}}{\operatorname{Im} g_k(t, z_k)}.$$

Hence, we obtain that

$$e^{\int_0^T |\zeta(s)| \mathrm{d}s} \leq \sqrt{1 + \frac{4T}{\delta^2}}.$$

This is the $C(T, \delta)$ that we wanted and we have

$$|\psi(T)| = |g_1(T, z_1) - g_2(T, z_2)| \le \sqrt{1 + \frac{4T}{\delta^2}} \left(\|W_1 - W_2\|_{\infty, [0,T]} + |z_1 - z_2| \right).$$

□

So we see that the Loewner map is a function between two measurable spaces equipped with Borel σ-algebras. By continuity, the Loewner map $L : (W_t)_{t\ge 0} \mapsto (g_t)_{t\ge 0}$ is measurable. We have shown that our definition of SLE(κ) makes sense measure-theoretically.

The Loewner-Kufarev theorem is a even stronger statement, which in particular states that the Loewner map L is a homeomorphism. See Theorem 8.5, [BN16].

5.2 First Properties

Consider $\kappa = 0$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z)}, \quad g_0(z) = z.$$

This is not a stochastic object. In fact, this describes the Loewner flow

$$g_t(z) = \sqrt{z^2 + 4t}, \quad [0, i \cdot 2\sqrt{t}].$$

From now on, we shall assume that $\kappa > 0$.

Theorem 12 (Theorem 5.1, [Kem17]). Let $(K_t)_{t\ge 0}$ be an SLE(κ) with $\kappa > 0$, and $(W_t)_{t\ge 0}$ the corresponding driving process. Let $(W_t)_{t\ge 0}$ be adapted to $(\mathcal{F}_t)_{t\ge 0}$. Then SLE(κ) satisfies the following properties.

- Scale invariance: For any $\lambda > 0$, $(\lambda K_{t/\lambda^2})_{t\ge 0} \overset{D}{=} (K_t)_{t\ge 0}$.
- Conformal Markov property: For any $s \in \mathbb{R}_{\ge 0}$, the family of hulls

  $$(\hat{K}_{s,t})_{t\ge 0} = (\overline{g_s(K_{s+t} \setminus K_s)} - W_s)_{t\ge 0}$$

  is independent of $\mathcal{F}_s$ and $(\hat{K}_{s,t})_{t\ge 0} \overset{D}{=} (K_t)_{t\ge 0}$.
- Strong conformal Markov property: For any a.s. finite stopping time τ wrt $(\mathcal{F}_t)_{t\ge 0}$, the family of hulls

  $$(\hat{K}_{\tau,t})_{t\ge 0} = (\overline{g_\tau(K_{\tau+t} \setminus K_\tau)} - W_\tau)_{t\ge 0}$$

  is independent of $\mathcal{F}_\tau$ and $(\hat{K}_{\tau,t})_{t\ge 0} \overset{D}{=} (K_t)_{t\ge 0}$.

Proof. **Scale invariance** We show that their respective mapping-out functions are identically distributed. Recall that the mapping-out function for $(\lambda K_{t/\lambda^2})_{t\ge 0}$ is

$$\lambda g_{t/\lambda^2}(z/\lambda) = \lambda g\left(\frac{t}{\lambda^2}, \frac{z}{\lambda}\right).$$

We rewrite the gs for clarity. If the driving function is identically distributed to $W(t)$, then we are done. Writing $W_\lambda(t)$ for the corresponding driving function, the Loewner equation is

$$\begin{aligned}
\partial_t\left(\lambda g\left(\frac{t}{\lambda^2},\frac{z}{\lambda}\right)\right) &= \frac{2}{\lambda g\left(\frac{t}{\lambda^2},\frac{z}{\lambda}\right)-W_\lambda(t)}, \quad g(0,z)=z,\\
\frac{1}{\lambda}(\partial_t g)\left(\frac{t}{\lambda^2},\frac{z}{\lambda}\right) &= \frac{2}{\lambda g\left(\frac{t}{\lambda^2},\frac{z}{\lambda}\right)-W_\lambda(t)},\\
(\partial_t g)\left(\frac{t}{\lambda^2},\frac{z}{\lambda}\right) &= \frac{2}{g\left(\frac{t}{\lambda^2},\frac{z}{\lambda}\right)-\frac{1}{\lambda}W_\lambda(t)},\\
\partial_t g(t,z) &= \frac{2}{g(t,z)-\frac{1}{\lambda}W_\lambda(\lambda^2 t)}.
\end{aligned}$$

Hence, we see that

$$\frac{1}{\lambda}W_\lambda(\lambda^2 t)\overset{D}{=}W(t)=\sqrt{\kappa}B(t)\overset{D}{=}\sqrt{\kappa}\cdot\frac{1}{\lambda}B(\lambda^2 t),$$

by the scale invariance of Brownian motion. So we conclude that $W_\lambda(t)\overset{D}{=}W(t)$.

Conformal Markov property Given an SLE(κ), $(K_t)_{t\geq 0}$, and some $s\geq 0$, we define the family of hulls

$$(\hat{K}_{s,t})_{t\geq 0}=(\overline{g_s(K_{s+t}\setminus K_s)}-W_s)_{t\geq 0}$$

which are associated with the following Loewner flow

$$(\hat{g}_{s,t}(z))_{t\geq 0}=(g_{s+t}\circ g_s^{-1}(z+W_s)-W_s)$$

From the Loewner equation,

$$\begin{aligned}
\frac{2}{\hat{g}_{s,t}(z)-\hat{W}_{s,t}} &= \partial_t\hat{g}_{s,t}(z),\\
&= (\partial_t g_{s+t})(g_s^{-1}(z+W_s)),\\
&= \frac{2}{g_{s+t}\circ g_s^{-1}(z+W_s)-W_{s+t}},\\
&= \frac{2}{\hat{g}_{s,t}(z)-(W_{s+t}-W_t)}.
\end{aligned}$$

So we have $\hat{W}_{s,t}=W_{s+t}-W_t$. Recalling the Markov property of Brownian motion, we have

$$\hat{W}_t=W_{s+t}-W_s\overset{D}{=}W_t$$

and that $W_{s+t}-W_s$ is independent of $\mathcal{F}_s$. Hence, we see that $(\hat{K}_{s,t})_{t\geq 0}\overset{D}{=}(K_t)_{t\geq 0}$ and is independent of $\mathcal{F}_s$.

Strong conformal Markov property The exact same, replacing s with τ and using the strong Markov property of Brownian motion instead of the usual Markov property. □

In fact, scale invariance and conformal Markov property characterises SLEs. Here, we provide a proof for the converse of the above theorem, without the strong conformal Markov property.

Theorem 13 (Theorem 9.1, [BN16]; Section 5.1.1, [Kem17])**.** If $(K_t)_{t\geq 0}$ is a $\mathcal{L}$-valued random variable which satisfies the scale invariance and conformal Markov property, then it is SLE(κ) for some $\kappa > 0$.

Proof. Fix $s \geq 0$. Define $\hat{K}_t = g_s(K_{s+t}) - W_s$ for all $t \geq 0$. The conformal Markov property implies that $(\hat{K}_t)_{t\geq 0}$ is distributed identically with $(K_t)_{t\geq 0}$, and independent of $\mathcal{F}_s$. The calculation involving $\hat{g}_{s,t}$ above gives

$$\hat{W}_{s,t} = W_{s+t} - W_s.$$

Since $(\hat{K}_t)_{t\geq 0}$ is independent of $\mathcal{F}_s$ by the conformal Markov property, $(W_{s+t} - W_s)_{t\geq 0}$ is independent of $\mathcal{F}_s$ and is distributed as $(W_t)_{t\geq 0}$. So $(W_t)_{t\geq 0}$ has independent and stationary increments. The Lévy-Khintchine Theorem implies that $(W_t)_{t\geq 0}$ is a Brownian motion with linear drift. That is, we have

$$W_t = KB(t) + \alpha t$$

for some $K > 0$, $\alpha \in \mathbb{R}$. Let $\kappa = K^2$. We want to show that $\alpha = 0$.

We need scale invariance for this. Let $\lambda > 0$ and consider $(\lambda K_{t/\lambda^2})_{t\geq 0}$. Scale invariance says that $(W_t^\lambda)_{t\geq 0}$ is distributed identically to $(W_t)_{t\geq 0}$. By a similar calculation as in the proof of the previous theorem, we have

$$W_t^{(\lambda)} = \lambda W_{t/\lambda^2}.$$

So we have

$$\lambda\sqrt{\kappa}B\left(\frac{t}{\lambda^2}\right) + \alpha\frac{t}{\lambda^2} \overset{D}{=} \sqrt{\kappa}B(t) + \alpha t,$$

$$\sqrt{\kappa}B(t) + \alpha\frac{t}{\lambda^2} \overset{D}{=} \sqrt{\kappa}B(t) + \alpha t.$$

This holds for all λ. Hence, $\alpha = 0$. $\square$

What is the general shape of SLE? This question will be touched upon more extensively when we are discussing its phases. However, based off the continuity of Brownian motion and hence that of the driving process, we note that as a stochastic Loewner chain, its instances must be of local growth. As we have seen before, hulls of local growth do not necessarily correspond to hulls that are generated by a curve (Figure 4.4(c), [Kem17]). However, we imagined the SLE to be a sort of random curve. How do we resolve this? Can we prove that the instances must be generated by a curve?

Theorem 14 (Rohde-Schramm, Lawler-Schramm-Werner; Theorem 9.2, [BN16]; Theorem 5.2, [Kem17])**.** Let $(K_t)_{t\geq 0}$ denote some SLE(κ) with $\kappa \geq 0$. We write $(g_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ for the corresponding Loewner flow and driving process. The map $g_t^{-1} : \mathbb{H} \to H_t$ extends continuously to $\overline{\mathbb{H}}$ for all $t \geq 0$, almost surely. Moreover, if we set $\gamma_t = g_t^{-1}(W_t)$, then $(\gamma_t)_{t\geq 0}$ is continuous and generates $(K_t)_{t\geq 0}$, almost surely.

We recall the shape of curves which generate families of hulls of local growth. The are self-intersecting but not self-crossing. Again refer to Figure 4.4, [Kem17].

We denote the pushforward measure of the SLE(κ) on the upper half-plane from 0 to ∞, which we have just defined, by $\mu^{(\mathbb{H},0,\infty)}$. We now extend this to all two-pointed domains (U, a, b): a simply connected domain U with two distinguished boundary points $a \neq b$.

Definition 9 (Definition 5.2, [Kem17]). Let $(K_t)_{t\geq 0}$ be a chordal SLE(κ) and (U, a, b) a two-pointed domain. We define the chordal SLE(κ) in U from a to b to be the image of $(K_t)_{t\geq 0}$ under any conformal onto map $\phi : \mathbb{H} \to U$ with $\phi(0) = a$ and $\phi(\infty) = b$. The corresponding measure is defined by $\mu^{(U,a,b)} = \mu^{(\mathbb{H},0,\infty)} \circ \phi^{-1}$.

Obviously, this is not well-defined if we say 'for any conformal onto map'. ϕ is not unique, and each ϕ incurs a time-change. Recall how $\mu^{(\mathbb{H},0,\infty)}$ is defined for $(K_t)_{t\geq 0}$ which are parametrised by the half-plane capacity. But this situation is not that bad. We notice that ϕ is unique only up to a *linear* time change. We can show this. Let $\phi, \psi : \mathbb{H} \to U$ be conformal isomorphisms such that $\phi(0) = \psi(0) = a, \phi(\infty) = \psi(\infty) = b$. Then we note that $\phi \circ \psi^{-1} \in \mathrm{Aut}(\mathbb{H})$ fixing 0 and ∞. Then it is a simple calculation to show that $\phi \circ \psi^{-1}(z) = cz$ for some $c > 0$.

What we can do is to pick a specific ϕ, called a scale of the SLE(κ) in U. Alternatively, we can just consider SLE(κ) in other simply connected domains to be a measure on the traces of curves.

We have previously parametrised by half-plane capacity. Why couldn't we do that now? Well, we are not in the upper half-plane, so it is not applicable. To fix this, we can use a different parametrisation by an intrinsic quantity that is conformally invariant under domain change: the Minkowski content. This is known as the natural parametrisation. See Lawler's upcoming book on SLE mentioned in the appendix.

6 Appendix

6.1 Online Materials

Courseware

- Course webpage for Curtis McMullen's *From Conformal Invariants to Percolation*. Accompanied by lecture videos on YouTube. The channel name is Koebe 1/4! https://people.math.harvard.edu/~ctm/home/text/class/harvard/219/21/html/index.html

- Jason Miller's course webpage for a Part III course on SLE at Cambridge. It appears to provide a minimal path to SLEs with only 46 pages (obviously with very few details). https://www.dpmms.cam.ac.uk/~jpm205/teaching/lent2019/index.html

- Notes on SLE by Nathanaël Berestycki (joint with James Norris). https://homepage.univie.ac.at/nathanael.berestycki/wp-content/uploads/2023/08/SLEvienna.pdf

- Norris's page for the notes and example sheets (exercises). This is an older version compared to the notes on Berestycki's page. https://www.statslab.cam.ac.uk/~james/Lectures/

- Apparently Lawler has another upcoming book on SLEs. https://www.math.uchicago.edu/~lawler/bookmaster.pdf

Other things

- Includes visualisations of some conformal automorphisms of the sphere. https://www.youtube.com/watch?v=1SiN9ttjgOE

- Schramm-Loewner evolution, SLE(4), and its Loewner flow., a visualisation of the action of the mapping out functions following SLE_4. https://www.youtube.com/watch?v=NkdlqW1hmlY

References

[Law05] Gregory F. Lawler. *Conformally invariant processes in the plane*. Mathematical surveys and monographs ; v. 114. Publication Title: Conformally invariant processes in the plane. Providence, R.I: American Mathematical Society, 2005. ISBN: 0-8218-3677-3.

[Mör+10] Peter. Mörters et al. *Brownian motion*. Cambridge series in statistical and probabilistic mathematics ; [30]. Cambridge, UK: Cambridge University Press, 2010. ISBN: 0-521-76018-6.

[BN16] Nathanaël Berestycki and James Norris. *Lectures on Schramm-Loewner Evolution*. Jan. 14, 2016. URL: https://www.statslab.cam.ac.uk/~james/Lectures/sle.pdf.

[Kem17] Antti. Kemppainen. *Schramm–Loewner Evolution*. 1st ed. 2017. SpringerBriefs in Mathematical Physics, 24. Publication Title: Schramm–Loewner Evolution. Cham: Springer International Publishing, 2017. ISBN: 3-319-65329-6.

[Bel20] Dmitry Beliaev. *Conformal maps and geometry*. Advanced textbooks in mathematics. Publication Title: Conformal maps and geometry. New Jersey ; World Scientific, 2020. ISBN: 978-1-78634-613-1.

[Kle20] Achim Klenke. *Probability Theory: A Comprehensive Course*. 3rd Edition 2020. Universitext. Cham: Springer Nature, 2020. ISBN: 0172-5939. DOI: 10.1007/978-3-030-56402-5.